ALMOST SURE FUNCTIONAL CENTRAL LIMIT THEOREM FOR NON-NESTLING RANDOM WALK IN RANDOM ENVIRONMENT

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ABSTRACT. We consider a non-nestling random walk in a product random environment. We assume an exponential moment for the step of the walk, uniformly in the environment. We prove an invariance principle (functional central limit theorem) under almost every environment for the centered and diffusively scaled walk. The main point behind the invariance principle is that the quenched mean of the walk behaves subdiffusively.

1. INTRODUCTION AND MAIN RESULT

We prove a quenched functional central limit theorem for non-nestling random walk in random environment (RWRE) on the d-dimensional integer lattice $\mathbb{Z}^d$ in dimensions $d \geq 2$. Here is a general description of the model, fairly standard since quite a while. An environment ω is a configuration of transition probability vectors $\omega = (\omega_x)_{x \in \mathbb{Z}^d} \in \Omega = \mathcal{P}^{\mathbb{Z}^d}$, where $\mathcal{P} = \{(p_z)_{z \in \mathbb{Z}^d} : p_z \geq 0, \sum_z p_z = 1\}$ is the simplex of all probability vectors on $\mathbb{Z}^d$. Vector $\omega_x = (\omega_{x,z})_{z \in \mathbb{Z}^d}$ gives the transition probabilities out of state x, denoted by $\pi_{x,y}(\omega) = \omega_{x,y-x}$. To run the random walk, fix an environment ω and an initial state $z \in \mathbb{Z}^d$. The random walk $X_{0,\infty} = (X_n)_{n \geq 0}$ in environment ω started at z is then the canonical Markov chain with state space $\mathbb{Z}^d$ whose path measure P_z^ω satisfies

$$P_z^\omega(X_0 = z) = 1 \quad \text{and} \quad P_z^\omega(X_{n+1} = y | X_n = x) = \pi_{x,y}(\omega).$$

On the space Ω we put its product σ-field $\mathfrak{S}$, natural shifts $\pi_{x,y}(T_z\omega) = \pi_{x+z,y+z}(\omega)$, and a $\{T_z\}$-invariant probability measure $\mathbb{P}$ that makes the system $(\Omega, \mathfrak{S}, (T_z)_{z \in \mathbb{Z}^d}, \mathbb{P})$ ergodic. In this paper $\mathbb{P}$ is an i.i.d. product measure on $\mathcal{P}^{\mathbb{Z}^d}$. In other words, the vectors $(\omega_x)_{x \in \mathbb{Z}^d}$ are i.i.d. across the sites x under $\mathbb{P}$.

Statements, probabilities and expectations under a fixed environment, such as the distribution P_z^ω above, are called *quenched*. When also the environment is averaged out, the notions are called *averaged*, or also *annealed*. In particular, the averaged distribution $P_z(dx_{0,\infty})$ of the walk is the marginal of the joint distribution $P_z(dx_{0,\infty}, d\omega) = P_z^\omega(dx_{0,\infty})\mathbb{P}(d\omega)$ on paths and environments.

Several excellent expositions on RWRE exist, and we refer the reader to the lectures [3], [15] and [18]. We turn to the specialized assumptions imposed on the model in this paper.

The main assumption is *non-nestling* (N) which guarantees a drift uniformly over the environments. The terminology was introduced by Zerner [19].

HYPOTHESIS (N). *There exists a vector $\hat{u} \in \mathbb{Z}^d \setminus \{0\}$ and a constant $\delta > 0$ such that*

$$\mathbb{P}\Big\{\omega : \sum_{z \in \mathbb{Z}^d} z \cdot \hat{u}\, \pi_{0,z}(\omega) \geq \delta\Big\} = 1.$$

There is no harm in assuming $\hat{u} \in \mathbb{Z}^d$, and this is convenient. We utilize two auxiliary assumptions: an exponential moment bound (M) on the steps of the walk, and some regularity (R) on the environments.

HYPOTHESIS (M). *There exist positive constants M and s_0 such that*

$$\mathbb{P}\Big\{\omega : \sum_{z \in \mathbb{Z}^d} e^{s_0|z|}\pi_{0,z}(\omega) \leq e^{s_0 M}\Big\} = 1.$$

HYPOTHESIS (R). *There exists a constant $\kappa > 0$ such that*

$$\mathbb{P}\Big\{\omega : \sum_{z:\, z\cdot\hat{u}=1} \pi_{0,z}(\omega) \geq \kappa \Big\} = 1. \tag{1.1}$$

Let $\mathcal{J} = \{z : \mathbb{E}\pi_{0,z} > 0\}$ be the set of admissible steps under $\mathbb{P}$. Then

$$\mathbb{P}\{\forall z : \pi_{0,0} + \pi_{0,z} < 1\} > 0 \quad and \quad \mathcal{J} \not\subset \mathbb{R}u \ \ for\ all\ u \in \mathbb{R}^d. \tag{1.2}$$

Assumption (1.1) above is stronger than needed. In the proofs it is actually used in the form (7.5) [Section 7] that permits backtracking before hitting the level $x \cdot \hat{u} = 1$. At the expense of additional technicalities in Section 7 quenched assumption (1.1) can be replaced by an averaged requirement.

Assumption (1.2) is used in Lemma 7.10. It is necessary for the quenched CLT as was discovered already in the simpler forbidden direction case we studied in [10] and [11]. Note that assumption (1.2) rules out the case $d = 1$. However, the issue is not whether the walk is genuinely d-dimensional, but whether the walk can explore its environment thoroughly enough to suppress the fluctuations of the quenched mean. Most work on RWRE takes uniform ellipticity and nearest-neighbor jumps as standing assumptions, which of course imply Hypotheses (M) and (R).

These assumptions are more than strong enough to imply a law of large numbers: there exists a velocity $v \neq 0$ such that

$$P_0\big\{ \lim_{n\to\infty} n^{-1}X_n = v \big\} = 1. \tag{1.3}$$

Representations for v are given in (2.5) and Lemma 5.1. Define the (approximately) centered and diffusively scaled process

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}}. \tag{1.4}$$

As usual $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ is the integer part of a real x. Let $D_{\mathbb{R}^d}[0,\infty)$ be the standard Skorohod space of $\mathbb{R}^d$-valued cadlag paths (see [6] for the basics). Let $Q_n^\omega = P_0^\omega(B_n \in \cdot)$ denote the quenched distribution of the process B_n on $D_{\mathbb{R}^d}[0,\infty)$.

The results of this paper concern the limit of the process B_n as $n \to \infty$. As expected, the limit process is a Brownian motion with correlated coordinates. For a symmetric, non-negative definite $d \times d$ matrix $\mathfrak{D}$, a *Brownian motion with diffusion matrix* $\mathfrak{D}$ is the $\mathbb{R}^d$-valued process $\{B(t) : t \geq 0\}$ with continuous paths, independent increments, and such that for $s < t$ the d-vector $B(t) - B(s)$ has Gaussian distribution with mean zero and covariance matrix $(t - s)\mathfrak{D}$. The matrix $\mathfrak{D}$ is *degenerate* in direction $u \in \mathbb{R}^d$ if $u^t \mathfrak{D} u = 0$. Equivalently, $u \cdot B(t) = 0$ almost surely.

Here is the main result.

THEOREM 1.1. *Let $d \geq 2$ and consider a random walk in an i.i.d. product random environment that satisfies non-nestling* (N), *the exponential moment hypothesis* (M), *and the regularity in* (R). *Then for $\mathbb{P}$-almost every ω distributions Q_n^ω converge weakly on $D_{\mathbb{R}^d}[0,\infty)$ to the distribution of a Brownian motion with a diffusion matrix $\mathfrak{D}$ that is independent of ω. $u^t \mathfrak{D} u = 0$ iff u is orthogonal to the span of $\{x - y : \mathbb{E}(\pi_{0x})\mathbb{E}(\pi_{0y}) > 0\}$.*

Eqn (2.6) gives the expression for the diffusion matrix $\mathfrak{D}$, familiar for example from [14]. Before turning to the proofs we discuss briefly the current situation in this area of probability and the place of this work in this context.

Several different approaches can be identified in recent work on quenched central limit theorems for multidimensional RWRE. (i) Small perturbations of classical random walk have been studied by many authors. The most significant results include the early work of Bricmont and Kupiainen [4] and more recently Sznitman and Zeitouni [16] for small perturbations of Brownian motion in dimension $d \geq 3$. (ii) An averaged CLT can be turned into a quenched CLT by bounding certain variances through the control of intersections of two independent paths. This idea was introduced by Bolthausen and Sznitman in [2] and more recently applied by Berger and Zeitouni in [1]. Both utilize high dimension to handle the intersections. (iii) Our approach is based on the subdiffusivity of the quenched mean of the walk. That is, we show that the variance of $E_0^\omega(X_n)$ is of order $n^{2\alpha}$ for some $\alpha < 1/2$. We also achieve this through intersection bounds. Instead of high dimension we assume strong enough

drift. We introduced this line of reasoning in [9] and later applied it to the case of walks with a forbidden direction in [11]. The significant advance taken in the present paper over [9] and [11] is the elimination of restrictions on the admissible steps of the walk. Theorem 2.1 below summarizes the general principle for application in this paper.

As the reader will see, the arguments in this paper are based on quenched exponential bounds that flow from Hypotheses (N), (M) and (R). It is common in this field to look for an invariant measure $\mathbb{P}_\infty$ for the environment process that is mutually absolutely continuous with the original $\mathbb{P}$, at least on the part of the space Ω to which the drift points. In this paper we do things a little differently: instead of the absolute continuity, we use bounds on the variation distance between $\mathbb{P}_\infty$ and $\mathbb{P}$. This distance will decay exponentially in the direction $\hat{u}$.

In the case of nearest-neighbor, uniformly elliptic non-nestling walks in dimension $d \geq 4$ the quenched CLT has been proved earlier: first by Bolthausen and Sznitman [2] under a small noise assumption, and recently by Berger and Zeitouni [1] without the small noise assumption. Berger and Zeitouni [1] go beyond non-nestling to more general ballistic walks. The method in these two papers utilizes high dimension crucially. Whether their argument can work in $d = 3$ is not presently clear. The approach of the present paper should work for more general ballistic walks in all dimensions $d \geq 2$, as the main technical step that reduces the variance estimate to an intersection estimate is generalized (Section 6 in the present paper).

We turn to the proofs. The next section collects some preliminary material and finishes with an outline of the rest of the paper.

2. PRELIMINARIES FOR THE PROOF.

As mentioned, we can assume that $\hat{u} \in \mathbb{Z}^d$. This is convenient because then the lattice $\mathbb{Z}^d$ decomposes into *levels* identified by the integer value $x \cdot \hat{u}$.

Let us summarize notation for the reader's convenience. Constants whose exact values are not important and can change from line to line are often denoted by C and s. The set of nonnegative integers is $\mathbb{N} = \{0, 1, 2, \dots\}$. Vectors and sequences are abbreviated $x_{m,n} = (x_m, x_{m+1}, \dots, x_n)$ and $x_{m,\infty} = (x_m, x_{m+1}, x_{m+2}, \dots)$. Similar notation is used for finite and infinite random paths: $X_{m,n} = (X_m, X_{m+1}, \dots, X_n)$ and $X_{m,\infty} = (X_m, X_{m+1}, X_{m+2}, \dots)$. $X_{[0,n]} = \{X_k : 0 \leq k \leq n\}$ denotes the set of sites visited by the walk. $\mathfrak{D}^t$ is the transpose of a vector or matrix $\mathfrak{D}$. An element of $\mathbb{R}^d$ is regarded as a $d \times 1$ column vector. The left shift on the path space $(\mathbb{Z}^d)^\mathbb{N}$ is $(\theta^k x_{0,\infty})_n = x_{n+k}$.

$\mathbb{E}$, E_0, and E_0^ω denote expectations under, respectively, $\mathbb{P}$, P_0, and P_0^ω. $\mathbb{P}_\infty$ will denote an invariant measure on Ω, with expectation $\mathbb{E}_\infty$. We abbreviate $P_0^\infty(\cdot) = \mathbb{E}_\infty P_0^\omega(\cdot)$ and $E_0^\infty(\cdot) = \mathbb{E}_\infty E_0^\omega(\cdot)$ to indicate that the environment of a quenched expectation is averaged under $\mathbb{P}_\infty$. A family of σ-algebras on Ω that in a sense look towards the future is defined by $\mathfrak{S}_\ell = \sigma\{\omega_x : x \cdot \hat{u} \geq \ell\}$.

Define the *drift*

$$D(\omega) = E_0^\omega(X_1) = \sum_z z\pi_{0z}(\omega).$$

The *environment process* is the Markov chain on Ω with transition kernel

$$\Pi(\omega, A) = P_0^\omega(T_{X_1}\omega \in A).$$

The proof of the quenched CLT Theorem 1.1 utilizes crucially the environment process and its invariant distribution. A preliminary part of the proof is summarized in the next theorem quoted from [9]. This Theorem 2.1 was proved by applying the arguments of Maxwell and Woodroofe [8] and Derriennic and Lin [5] to the environment process.

THEOREM 2.1. [9] *Let* $d \geq 1$. *Suppose the probability measure* $\mathbb{P}_\infty$ *on* $(\Omega, \mathfrak{S})$ *is invariant and ergodic for the Markov transition* Π. *Assume that* $\sum_z |z|^2\mathbb{E}_\infty(\pi_{0z}) < \infty$ *and that there exists an* $\alpha < 1/2$ *such that as* $n \to \infty$

$$\mathbb{E}_\infty\left[\,|E_0^\omega(X_n) - n\mathbb{E}_\infty(D)|^2\,\right] = O(n^{2\alpha}). \tag{2.1}$$

Then as $n \to \infty$ *the following weak limit happens for* $\mathbb{P}_\infty$*-a.e.* ω: *distributions* Q_n^ω *converge weakly on the space* $D_{\mathbb{R}^d}[0, \infty)$ *to the distribution of a Brownian motion with a symmetric, non-negative definite diffusion matrix* $\mathfrak{D}$ *that is independent of* ω.

Another central tool for the development that follows is provided by the *Sznitman-Zerner regeneration times* [17] that we now define. For $\ell \geq 0$ let λ_ℓ be the first time the walk reaches level ℓ relative to the initial level:

$$\lambda_\ell = \min\{n \geq 0 : X_n \cdot \hat{u} - X_0 \cdot \hat{u} \geq \ell\}.$$

Define β to be the first backtracking time:

$$\beta = \inf\{n \geq 0 : X_n \cdot \hat{u} < X_0 \cdot \hat{u}\}.$$

Let M_n be the maximum level, relative to the starting level, reached by time n:

$$M_n = \max\{X_k \cdot \hat{u} - X_0 \cdot \hat{u} : 0 \leq k \leq n\}.$$

For $a > 0$, and when $\beta < \infty$, consider the first time by which the walker reaches level $M_\beta + a$:

$$\lambda_{M_\beta+a} = \inf\{n \geq \beta : X_n \cdot \hat{u} - X_0 \cdot \hat{u} \geq M_\beta + a\}.$$

Let $S_0 = \lambda_a$ and, as long as $\beta \circ \theta^{S_{k-1}} < \infty$, define $S_k = S_{k-1} + \lambda_{M_\beta+a} \circ \theta^{S_{k-1}}$ for $k \geq 1$. Finally, let the first regeneration time be

$$\tau_1^{(a)} = \sum_{\ell \geq 0} S_\ell \mathbb{I}\{\beta \circ \theta^{S_k} < \infty \text{ for } 0 \leq k < \ell \text{ and } \beta \circ \theta^{S_\ell} = \infty\}. \tag{2.2}$$

Non-nestling guarantees that $\tau_1^{(a)}$ is finite, and in fact gives moment bounds uniformly in ω as we see in Lemma 3.1 below. Consequently we can iterate to define $\tau_0^{(a)} = 0$, and for $k \geq 1$

$$\tau_k^{(a)} = \tau_{k-1}^{(a)} + \tau_1^{(a)} \circ \theta^{\tau_{k-1}^{(a)}}. \tag{2.3}$$

When the value of a is not important we simplify the notation to $\tau_k = \tau_k^{(a)}$. Sznitman and Zerner [17] proved that the *regeneration slabs*

$$
\begin{aligned}
\mathcal{S}_k = \big(& \tau_{k+1} - \tau_k, \, (X_{\tau_k+n} - X_{\tau_k})_{0 \le n \le \tau_{k+1}-\tau_k}, \\
& \{\omega_{X_{\tau_k}+z} : 0 \le z \cdot \hat{u} < (X_{\tau_{k+1}} - X_{\tau_k}) \cdot \hat{u}\} \big)
\end{aligned}
\tag{2.4}
$$

are i.i.d. for $k \ge 1$, each distributed as $\big(\tau_1, \, (X_n)_{0 \le n \le \tau_1}, \, \{\omega_z : 0 \le z \cdot \hat{u} < X_{\tau_1} \cdot \hat{u}\} \big)$ under $P_0(\,\cdot\,|\,\beta = \infty)$. Strictly speaking, uniform ellipticity and nearest-neighbor jumps were standing assumptions in [17], but these assumptions are not needed for the proof of the i.i.d. structure.

From the renewal structure and moment estimates a law of large numbers (1.3) and an averaged functional central limit theorem follow, along the lines of Theorem 2.3 in [17] and Theorem 4.1 in [14]. These references treat walks that satisfy Kalikow's condition, considerably more general than the non-nestling walks we study. The limiting velocity for the law of large numbers is

$$
v = \frac{E_0(X_{\tau_1}|\beta = \infty)}{E_0(\tau_1|\beta = \infty)}.
\tag{2.5}
$$

The averaged CLT states that the distributions $P_0\{B_n \in \,\cdot\,\}$ converge to the distribution of a Brownian motion with diffusion matrix

$$
\mathfrak{D} = \frac{E_0\big[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t | \beta = \infty\big]}{E_0[\tau_1|\beta = \infty]}.
\tag{2.6}
$$

Once we know that the $\mathbb{P}$-a.s. quenched CLT holds with a constant diffusion matrix, this diffusion matrix must be the same $\mathfrak{D}$ as for the averaged CLT. We give here the argument for the degeneracy statement of Theorem 1.1.

LEMMA 2.1. *Define $\mathfrak{D}$ by* (2.6) *and let $u \in \mathbb{R}^d$. Then $u^t \mathfrak{D} u = 0$ iff u is orthogonal to the span of $\{x - y : \mathbb{E}(\pi_{0x})\mathbb{E}(\pi_{0y}) > 0\}$.*

Proof. The argument is a minor embellishment of that given for a similar degeneracy statement on p. 123–124 of [10] for the forbidden-direction case where $\pi_{0,z}$ is supported by $z \cdot \hat{u} \ge 0$. We spell out enough of the argument to show how to adapt that proof to the present case.

Again, the intermediate step is to show that $u^t \mathfrak{D} u = 0$ iff u is orthogonal to the span of $\{x - v : \mathbb{E}(\pi_{0x}) > 0\}$. The argument from orthogonality to $u^t \mathfrak{D} u = 0$ goes as in [10, p. 124].

Suppose $u^t \mathfrak{D} u = 0$ which is the same as

$$
P_0(X_{\tau_1} \cdot u = \tau_1 v \cdot u \,|\, \beta = \infty) = 1.
$$

Suppose z is such that $\mathbb{E}\pi_{0,z} > 0$ and $z \cdot \hat{u} < 0$. By non-nestling there must exist w such that $\mathbb{E}\pi_{0,z}\pi_{0,w} > 0$ and $w \cdot \hat{u} > 0$. Pick $m > 0$ so that $(z + mw) \cdot \hat{u} > 0$ but

$(z + (m-1)w) \cdot \hat{u} \le 0$. Take $a = 1$ in the definition (2.2) of regeneration. Then

$$P_0[X_{\tau_1} = z + 2mw, \tau_1 = 2m+1 \,|\, \beta = \infty]$$

$$\ge \mathbb{E}\Big[\Big(\prod_{i=0}^{m-1} \pi_{iw,(i+1)w} \Big) \pi_{mw,z+mw} \Big(\prod_{j=0}^{m-1} \pi_{z+(m+j)w,z+(m+j+1)w} \Big) P_{z+2mw}^{\omega}(\beta = \infty) \Big] > 0.$$

Consequently

$$(z + 2mw) \cdot u = (1 + 2m)v \cdot u. \tag{2.7}$$

In this manner, by replacing σ_1 with τ_1 and by adding in the no-backtracking probabilities, the arguments in [10, p. 123] can be repeated to show that if $\mathbb{E}\pi_{0x} > 0$ then $x \cdot u = v \cdot u$ for x such that $x \cdot \hat{u} \ge 0$. In particular the very first step on p. 123 of [10] gives $w \cdot u = v \cdot u$. This combines with (2.7) above to give $z \cdot u = v \cdot u$. Now simply follow the proof in [10, p. 123–124] to its conclusion. $\square$

Here is an outline of the proof of Theorem 1.1. It all goes via Theorem 2.1.

(i) After some basic estimates in Section 3, we prove in Section 4 the existence of the ergodic equilibrium $\mathbb{P}_{\infty}$ required for Theorem 2.1. $\mathbb{P}_{\infty}$ is not convenient to work with so we still need to do computations with $\mathbb{P}$. For this purpose Section 4 proves that in the direction $\hat{u}$ the measures $\mathbb{P}_{\infty}$ and $\mathbb{P}$ come exponentially close in variation distance and that the environment process satisfies a P_0-a.s. ergodic theorem. In Section 5 we show that $\mathbb{P}_{\infty}$ and $\mathbb{P}$ are interchangeable both in the hypotheses that need to be checked and in the conclusions obtained. In particular, the $\mathbb{P}_{\infty}$-a.s. quenched CLT coming from Theorem 2.1 holds also $\mathbb{P}$-a.s. Then we know that the diffusion matrix $\mathfrak{D}$ is the one in (2.6).

The bulk of the work goes towards verifying condition (2.1), but under $\mathbb{P}$ instead of $\mathbb{P}_{\infty}$. There are two main stages to this argument.

(ii) By a decomposition into martingale increments the proof of (2.1) reduces to bounding the number of common points of two independent walks in a common environment (Section 6).

(iii) The intersections are controlled by introducing levels at which both walks regenerate. These common regeneration levels are reached fast enough and the progression from one common regeneration level to the next is a Markov chain. When this Markov chain drifts away from the origin it can be approximated well enough by a symmetric random walk. This approximation enables us to control the growth of the Green function of the Markov chain, and thereby the number of common points. This is in Section 7 and in an Appendix devoted to the Green function bound.

3. Basic estimates for non-nestling RWRE

This section contains estimates that follow from Hypotheses (N) and (M), all collected in the following lemma. These will be used repeatedly. In addition to the

stopping times already defined, let

$$H_z = \min\{n \geq 1 : X_n = z\}$$

be the first hitting time of site z.

LEMMA 3.1. *If $\mathbb{P}$ satisfies Hypotheses (N) and (M), then there exist positive constants η, γ, κ, $(C_p)_{p \geq 1}$, and $s_1 \leq s_0$, possibly depending on M, s_0, and δ, such that for all $x \in \mathbb{Z}^d$, $n \geq 0$, $s \in [0, s_1]$, $p \geq 1$, $\ell \geq 1$, for z such that $z \cdot \hat{u} \geq 0$, $a \geq 1$, and for $\mathbb{P}$-a.e. ω,*

$$E_x^\omega(e^{-sX_n \cdot \hat{u}}) \leq e^{-sx \cdot \hat{u}}(1 - s\delta/2)^n, \tag{3.1}$$

$$E_x^\omega(e^{s|X_n - x|}) \leq e^{sMn}, \tag{3.2}$$

$$P_x^\omega(X_1 \cdot \hat{u} \geq x \cdot \hat{u} + \gamma) \geq \kappa, \tag{3.3}$$

$$E_x^\omega(\lambda_\ell^p) \leq C_p \ell^p, \tag{3.4}$$

$$E_x^\omega(|X_{\lambda_\ell} - x|^p) \leq C_p \ell^p, \tag{3.5}$$

$$E_0^\omega[(M_{H_z} - z \cdot \hat{u})^p \mathbb{I}\{H_z < n\}] \leq C_p \ell^p P_0^\omega(H_z < n) + C_p s^{-p} e^{-s\ell/2}, \tag{3.6}$$

$$P_x^\omega(\beta = \infty) \geq \eta, \tag{3.7}$$

$$E_x^\omega(|\tau_1^{(a)}|^p) \leq C_p a^p, \tag{3.8}$$

$$E_x^\omega(|X_{\tau_1^{(a)}+n} - X_n|^p) \leq C_q a^q, \text{ for all } q > p. \tag{3.9}$$

The particular point in (3.8)–(3.9) is to make the dependence on a explicit. Note that (3.7)–(3.8) give

$$E_0(\tau_j - \tau_{j-1})^p < \infty \tag{3.10}$$

for all $j \geq 1$. In Section 4 we construct an ergodic invariant measure $\mathbb{P}_\infty$ for the environment chain in a way that preserves the conclusions of this lemma under $\mathbb{P}_\infty$.

Proof. Replacing x by 0 and ω by $T_x\omega$ allows us to assume that $x = 0$. Then for all $s \in [0, s_0/2]$

$$\left| E_0^\omega(e^{-sX_1 \cdot \hat{u}}) - 1 + sE_0^\omega(X_1 \cdot \hat{u}) \right| \leq |\hat{u}|^2 E_0^\omega(|X_1|^2 e^{s_0|X_1|/2}) \frac{s^2}{2}$$
$$\leq (2|\hat{u}|/s_0)^2 e^{s_0 M} s^2 = cs^2,$$

where we used moment assumption (M). Then by the non-nestling assumption (N)

$$E_0^\omega(e^{-sX_n \cdot \hat{u}}|X_{n-1}) = e^{-sX_{n-1} \cdot \hat{u}} E_{X_{n-1}}^\omega(e^{-s(X_1 - X_0) \cdot \hat{u}}) \leq e^{-sX_{n-1} \cdot \hat{u}}(1 - s\delta + cs^2).$$

Taking now the quenched expectation of both sides and iterating the procedure proves (3.1), provided s_1 is small enough. To prove (3.2) one can instead show that

$$E_0^\omega(e^{s\sum_{k=1}^n |X_k - X_{k-1}|}) \leq e^{snM}.$$

This can be proved by induction as for (3.1), using only Hypothesis (M) and Hölder's inequality (to switch to s_0).

Concerning (3.3), we have

$$P_0^\omega(X_1 \cdot \hat{u} \geq \gamma) \geq (1 - e^{\gamma s}(1 - s\delta/2)) \xrightarrow[\gamma \to 0]{} s\delta/2.$$

So taking γ small enough and κ slightly smaller than $s\delta/2$ does the job.

Notice next that $P_0^\omega(\lambda_1 < \infty) = 1$ due to (3.1). $\mathbb{P}$-a.s. Then

$$E_0^\omega(\lambda_1^p) \leq \sum_{n \geq 0}(n+1)^p P_0^\omega(\lambda_1 > n) \leq \sum_{n \geq 0}(n+1)^p P_0^\omega(X_n \cdot \hat{u} \leq 1)$$

$$\leq e^s \sum_{n \geq 0}(n+1)^p E_0^\omega(e^{-sX_n \cdot \hat{u}}).$$

The last expression is bounded if s is small enough. Therefore,

$$E_0^\omega(\lambda_\ell^p) \leq E_0^\omega\left[\left|\sum_{i=1}^{[\ell]+1}(\lambda_i - \lambda_{i-1})\right|^p\right] \leq ([\ell]+1)^{p-1}\sum_{i=1}^{[\ell]+1}E_0^\omega\left[E_{X_{\lambda_{i-1}}}^\omega(\lambda_1^p)\right] \leq C_p\ell^p.$$

Bound (3.5) is proved similarly: by the Cauchy-Schwarz inequality, Hypothesis (M) and (3.1),

$$E_0^\omega(|X_{\lambda_1}|^p) \leq \sum_{n \geq 1}E_0^\omega(|X_n|^{2p})^{1/2}P_0^\omega(X_{n-1} \cdot \hat{u} < 1)^{1/2}$$

$$\leq \left([2p]! \, s_0^{-[2p]}e^{s_0 M}e^s\right)^{1/2}\sum_{n \geq 1}(1 - s\delta/2)^{(n-1)/2}n^p \leq C_p.$$

To prove (3.6), write

$$E_0^\omega[(M_{H_z} - z \cdot \hat{u})^p \mathbb{I}\{H_z < n\}]$$

$$\leq C_p \sum_{\ell > \ell_0}\ell^{p-1}P_0^\omega(M_{H_z} - z \cdot \hat{u} \geq \ell, H_z < n) + C_p\ell_0^p P_0^\omega(H_z < n)$$

$$\leq C_p \sum_{\ell > \ell_0}\sum_{k \geq 0}\ell^{p-1}E_0^\omega[P_{X_{\lambda_{z \cdot \hat{u}+\ell}}}^\omega(X_k \cdot \hat{u} - X_0 \cdot \hat{u} \leq -\ell)] + C_p\ell_0^p P_0^\omega(H_z < n)$$

$$\leq C_p \sum_{\ell > \ell_0}\ell^{p-1}e^{-s\ell} + C_p\ell_0^p P_0^\omega(H_z < n) \leq C_p s^{-p}e^{-s\ell_0/2} + C_p\ell_0^p P_0^\omega(H_z < n).$$

To prove (3.7), note that Chebyshev inequality and (3.1) give, for $s > 0$ small enough, $\ell \geq 1$, and $\mathbb{P}$-a.e. ω

$$P_0^\omega(\lambda_{-\ell+1} < \infty) \leq \sum_{n \geq 0}P_0^\omega(X_n \cdot \hat{u} \leq -(\ell-1)) \leq 2(s\delta)^{-1}e^{-s(\ell-1)}.$$

On the other hand, for an integer $\ell \geq 2$ we have

$$P_0^\omega(\lambda_\ell < \beta) \geq \sum_x P_0^\omega(\lambda_{\ell-1} < \beta, X_{\lambda_{\ell-1}} = x)P_x^\omega(\lambda_{-\ell+1} = \infty).$$

Therefore, taking ℓ to infinity one has, for ℓ_0 large enough,

$$P_0^\omega(\beta = \infty) \geq P_0^\omega(\lambda_{\ell_0} < \beta) \prod_{\ell \geq \ell_0}(1 - 2(s\delta)^{-1}e^{-s\ell}).$$

Markov property and (3.3) give $P_0^\omega(\lambda_{\ell_0} < \beta) \geq \kappa^{\ell_0/\gamma+1} > 0$ and (3.7) is proved.

Now we will bound the quenched expectation of $\lambda_{M_\beta+a}^p \mathbb{I}\{\beta < \infty\}$ uniformly in ω. To this end, for $p_1 > p$ and $q_1 = p_1(p_1 - p)^{-1}$, we have by (3.7)

$$E_0^\omega(\lambda_{M_\beta+a}^p \mathbb{I}\{\beta < \infty\}) \leq \sum_{n \geq 1} E_0^\omega(\lambda_{M_n+a}^p \mathbb{I}\{\beta = n\})$$

$$\leq \sum_{n \geq 1} \left(E_0^\omega(\lambda_{M_n+a}^{p_1})\right)^{p/p_1} \left(P_0^\omega(\beta = n)\right)^{1/q_1}.$$

By (3.4) one has, for $p_2 > p_1 > p$ and $q_2 = p_2(p_2 - p_1)^{-1}$,

$$E_0^\omega(\lambda_{M_n+a}^{p_1}) \leq \sum_{m \geq 0} \left(E_0^\omega(\lambda_{m+1+a}^{p_2})\right)^{p_1/p_2} \left(P_0^\omega([M_n] = m)\right)^{1/q_2}$$

$$\leq C_p \sum_{m \geq 0} (m + 1 + a)^{p_1} \left(\sum_{i=0}^n P_0^\omega(X_i \cdot \hat{u} \geq m)\right)^{1/q_2},$$

where C_p really depends on p_1 and p_2, but these are chosen arbitrarily, as long as they satisfy $p_2 > p_1 > p$. Using (3.2) one has

$$P_0^\omega(X_i \cdot \hat{u} \geq m) \leq \begin{cases} 1 & \text{if } m < 2M|\hat{u}|i, \\ e^{-sm}e^{M|\hat{u}|si} & \text{if } m \geq 2M|\hat{u}|i. \end{cases}$$

Hence,

$$E_0^\omega(\lambda_{M_n+a}^{p_1}) \leq C_p \sum_{m \geq 0} (m + 1 + a)^{p_1} (n\mathbb{I}\{m < 2Mn|\hat{u}|\} + e^{-sm/2})^{1/q_2}$$

$$\leq C_p n(n + a)^{p_1} n^{1/q_2} + C_p \sum_{m \geq 0} (m + 1)^{p_1} e^{-sm/2q_2} + C_p a^{p_1} \sum_{m \geq 0} e^{-sm/2q_2}$$

$$\leq C_p n^{1+1/q_2}(n + a)^{p_1}.$$

Since $\{\beta = n\} \subset \{X_n \cdot \hat{u} \leq 0\}$, one can use (3.1) to conclude that

$$E_0^\omega(\lambda_{M_\beta+a}^p \mathbb{I}\{\beta < \infty\}) \leq C_p \sum_{n \geq 1} n^{p/p_1+p/(p_1 q_2)}(n + a)^p(1 - s\delta/2)^{n/q_1} \leq C_p a^p.$$

In the last inequality we have used the fact that $a \geq 1$. Using, (3.7), the definition of the times S_k, and the Markov property, one has

$$E_0^\omega[S_\ell^p \mathbb{1}\{\beta \circ \theta^{S_k} < \infty \text{ for } 0 \leq k < \ell \text{ and } \beta \circ \theta^{S_\ell} = \infty\}]$$

$$\leq (\ell+1)^{p-1} \Big(E_0^\omega[\lambda_a^p \mathbb{1}\{\beta \circ \theta^{S_k} < \infty \text{ for } 0 \leq k < \ell\}]$$

$$+ \sum_{j=0}^{\ell-1} E_0^\omega[\lambda_{M_\beta+a}^p \circ \theta^{S_j} \mathbb{1}\{\beta \circ \theta^{S_k} < \infty \text{ for } 0 \leq k < \ell\}]\Big)$$

$$\leq (\ell+1)^{p-1}\Big(C_p a^p (1-\eta)^\ell + \sum_{j=0}^{\ell-1}(1-\eta)^j C_p a^p (1-\eta)^{\ell-j-1}\Big)$$

$$\leq C_p(\ell+1)^p(1-\eta)^{\ell-1}a^p.$$

Bound (3.8) follows then from (2.2). To prove (3.9) let $q > p$ and write

$$E_0^\omega(|X_{\tau_1^{(a)}+n} - X_n|^p) \leq \sum_{k\geq 0} E_0^\omega(|X_{k+1+n} - X_{k+n}|^p |\tau_1^{(a)}|^{p-1} \mathbb{1}\{k < \tau_1^{(a)}\})$$

$$\leq \sum_{k\geq 0} k^{-1-q+p} E_0^\omega(|X_{k+1+n} - X_{k+n}|^p |\tau_1^{(a)}|^q)$$

$$\leq C_p \sum_{k\geq 0} k^{-1-q+p} E_0^\omega(|\tau_1^{(a)}|^{2q})^{1/2} \leq C_q\, a^q,$$

where we have used Hypothesis (M) along with the Cauchy-Schwarz inequality in the second to last inequality and (3.8) in the last. This completes the proof of the lemma. $\qquad\square$

4. Invariant measure and ergodicity

For $\ell \in \mathbb{Z}$ define the σ-algebras $\mathfrak{S}_\ell = \sigma\{\omega_x : x \cdot \hat{u} \geq \ell\}$ on Ω. Denote the restriction of the measure $\mathbb{P}$ to the σ-algebra $\mathfrak{S}_\ell$ by $\mathbb{P}_{|\mathfrak{S}_\ell}$. In this section we prove the next two theorems. The variation distance of two probability measures is $d_{\mathrm{Var}}(\mu,\nu) = \sup\{\mu(A) - \nu(A)\}$ with the supremum taken over measurable sets A.

THEOREM 4.1. *Assume $\mathbb{P}$ is product non-nestling* (N) *and satisfies the moment hypothesis* (M). *Then there exists a probability measure $\mathbb{P}_\infty$ on Ω with these properties.*
  (a) *$\mathbb{P}_\infty$ is invariant and ergodic for the Markov transition kernel Π.*
  (b) *There exist constants $0 < c, C < \infty$ such that for all $\ell \geq 0$*
$$d_{\mathrm{Var}}(\mathbb{P}_{\infty|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \leq Ce^{-c\ell}. \tag{4.1}$$
  (c) *Hypotheses* (N) *and* (M) *and the conclusions of Lemma* 3.1 *hold $\mathbb{P}_\infty$-almost surely.*

Along the way we also establish this ergodic theorem under the original environment measure. $\mathbb{E}_\infty$ denotes expectation under $\mathbb{P}_\infty$.

THEOREM 4.2. *Assumptions as in Theorem 4.1 above. Let Ψ be a bounded $\mathfrak{S}_{-a}$-measurable function on Ω, for some $0 < a < \infty$. Then*

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} \Psi(T_{X_j} \omega) = \mathbb{E}_\infty \Psi \quad P_0\text{-almost surely.} \tag{4.2}$$

The ergodic theorem tells us that there is a unique invariant $\mathbb{P}_\infty$ in a natural relationship to $\mathbb{P}$, and that $\mathbb{P}_\infty \ll \mathbb{P}$ on each σ-algebra $\mathfrak{S}_{-a}$. Limit (4.2) cannot hold for all bounded measurable Ψ on Ω because this would imply the absolute continuity $\mathbb{P}_\infty \ll \mathbb{P}$ on the entire space Ω. A counterexample that satisfies (N) and (M) but where the quenched walk is degenerate was given by Bolthausen and Sznitman [2, Proposition 1.5]. Whether regularity assumption (R) or ellipticity will make a difference here is not presently clear. For the simpler case of space-time walks (see description of model in [9]) with nondegenerate P_0^ω absolute continuity $\mathbb{P}_\infty \ll \mathbb{P}$ does hold on the entire space. Theorem 3.1 in [2] proves this for nearest-neighbor jumps with some weak ellipticity. The general case is no harder.

Proof of Theorems 4.1 and 4.2. Let $\mathbb{P}_n(A) = P_0(T_{X_n} \omega \in A)$. A computation shows that

$$f_n(\omega) = \frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega) = \sum_x P_x^\omega(X_n = 0).$$

By hypotheses (M) and (N) we can replace the state space $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ with the smaller space $\Omega_0 = \mathcal{P}_0^{\mathbb{Z}^d}$ where

$$\mathcal{P}_0 = \{(p_z) \in \mathcal{P} : \sum_z e^{s_0|z|} p_z \le e^{s_0 M} \text{ and } \sum_z z \cdot \hat{u}\, p_z \ge \delta \}. \tag{4.3}$$

Fatou's lemma shows that the exponential bound is preserved by pointwise convergence in $\mathcal{P}_0$. Then the exponential bound shows that the non-nestling property is also preserved. Thus $\mathcal{P}_0$ is compact, and then Ω_0 is compact under the product topology.

Compactness gives a subsequence $\{n_j\}$ along which the averages $n_j^{-1} \sum_{m=1}^{n_j} \mathbb{P}_m$ converge weakly to a probability measure $\mathbb{P}_\infty$ on Ω_0. Hypotheses (N) and (M) transfer to $\mathbb{P}_\infty$ by virtue of having been included in the state space Ω_0. Thus the proof of Lemma 3.1 can be repeated for $\mathbb{P}_\infty$-a.e. ω. We have verified part (c) of Theorem 4.1.

Next we check that $\mathbb{P}_\infty$ is invariant under Π. Take a bounded, continuous local function F on Ω_0 that depends only on environments $(\omega_x : |x| \le K)$. For $\omega, \bar{\omega} \in \Omega_0$

$$\left| \Pi F(\omega) - \Pi F(\bar{\omega}) \right| = \left| E_0^\omega[F(T_{X_1}\omega)] - E_0^{\bar{\omega}}[F(T_{X_1}\bar{\omega})] \right|$$

$$\le \sum_{|z| \le C} \left| \pi_{0,z}(\omega)F(T_z\omega) - \pi_{0,z}(\bar{\omega})F(T_z\bar{\omega}) \right| + \|F\|_\infty \sum_{|z| > C} \left(\pi_{0,z}(\omega) + \pi_{0,z}(\bar{\omega}) \right).$$

From this we see that ΠF is continuous. For let $\bar{\omega} \to \omega$ in Ω_0 so that $\bar{\omega}_{x,z} \to \omega_{x,z}$ at each coordinate. Since the last term above is controlled by the uniform exponential

tail bound imposed on $\mathcal{P}_0$, continuity of ΠF follows. Consequently the weak limit $n_j^{-1} \sum_{m=1}^{n_j} \mathbb{P}_m \to \mathbb{P}_\infty$ together with $\mathbb{P}_{n+1} = \mathbb{P}_n \Pi$ implies the Π-invariance of $\mathbb{P}_\infty$.

We show the exponential bound (4.1) on the variation distance next because the ergodicity proof depends on it. On metric spaces total variation distance can be characterized in terms of continuous functions:

$$d_{\mathrm{Var}}(\mu, \nu) = \frac{1}{2} \sup \left\{ \int f d\mu - \int f d\nu : f \text{ continuous, } \sup |f| \le 1 \right\}.$$

This makes $d_{\mathrm{Var}}(\mu, \nu)$ lower semicontinuous which we shall find convenient below.

Fix $\ell > 0$. Then

$$\begin{aligned} \frac{d\mathbb{P}_{n|\mathfrak{S}_\ell}}{d\mathbb{P}_{|\mathfrak{S}_\ell}} &= \mathbb{E}\Big[\sum_x P_x^\omega(X_n = 0, \max_{j \le n} X_j \cdot \hat{u} \le \ell/2) \big| \mathfrak{S}_\ell \Big] \\ &\quad + \sum_x \mathbb{E}[P_x^\omega(X_n = 0, \max_{j \le n} X_j \cdot \hat{u} > \ell/2) | \mathfrak{S}_\ell]. \end{aligned} \tag{4.4}$$

The $L^1(\mathbb{P})$-norm of the second term is bounded by

$$I_{n,\ell} = P_0(\max_{j \le n} X_j \cdot \hat{u} > X_n \cdot \hat{u} + \ell/2)$$

and (3.1) tells us that

$$I_{n,\ell} \le \sum_{j=0}^n e^{-s\ell/2}(1 - s\delta/2)^{n-j} \le Ce^{-s\ell/2}. \tag{4.5}$$

The integrand in the first term of (4.4) is measurable with respect to $\sigma(\omega_x : x \cdot \hat{u} \le \ell/2)$ and therefore independent of $\mathfrak{S}_\ell$. The distance between the whole first term and 1 is then $O(I_{n,\ell})$. Thus for large enough ℓ,

$$d_{\mathrm{Var}}(\mathbb{P}_{n|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \le \int \Big| \frac{d\mathbb{P}_{n|\mathfrak{S}_\ell}}{d\mathbb{P}_{|\mathfrak{S}_\ell}} - 1 \Big| d\mathbb{P} \le 2I_{n,\ell} \le Ce^{-c\ell}.$$

By the construction of $\mathbb{P}_\infty$ as the Cesàro limit and by the lower semicontinuity and convexity of the variation distance

$$d_{\mathrm{Var}}(\mathbb{P}_{\infty|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \le \varliminf_{j \to \infty} n_j^{-1} \sum_{m=1}^{n_j} d_{\mathrm{Var}}(\mathbb{P}_{m|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \le Ce^{-c\ell}.$$

Part (b) has been verified.

As the last point we prove the ergodicity. Recall the notation $E_0^\infty = \mathbb{E}_\infty E_0^\omega$. Let Ψ be a bounded local function on Ω. It suffices to prove that for some constant b

$$\lim_{n \to \infty} E_0^\infty \Big| n^{-1} \sum_{j=0}^{n-1} \Psi(T_{X_j}\omega) - b \Big| = 0. \tag{4.6}$$

By an approximation it follows from this that for all $F \in L^1(\mathbb{P}_\infty)$

$$n^{-1} \sum_{j=0}^{n-1} \Pi^j F(\omega) \to \mathbb{E}_\infty F \quad \text{in } L^1(\mathbb{P}_\infty). \tag{4.7}$$

By standard theory (Section IV.2 in [12]) this is equivalent to ergodicity of $\mathbb{P}_\infty$ for the transition Π.

We combine the proof of Theorem 4.2 with the proof of (4.6). For this purpose let Ψ be $\mathfrak{S}_{-a+1}$-measurable with $a < \infty$. Take a to be the parameter in the regeneration times (2.2). Let

$$\varphi_i = \sum_{j=\tau_i}^{\tau_{i+1}-1} \Psi(T_{X_j}\omega).$$

From the i.i.d. regeneration slabs and the moment bound (3.10) follows the limit

$$\lim_{m\to\infty} m^{-1} \sum_{j=0}^{\tau_m-1} \Psi(T_{X_j}\omega) = \lim_{m\to\infty} m^{-1} \sum_{i=0}^{m-1} \varphi_i = b_0 \qquad P_0\text{-almost surely}, \tag{4.8}$$

where the constant b_0 is defined by the limit.

To justify this more precisely, recall the definition of regeneration slabs given in (2.4). Define a function Φ of the regeneration slabs by

$$\Phi(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots) = \sum_{j=\tau_1}^{\tau_2-1} \Psi(T_{X_j}\omega).$$

Since each regeneration slab has thickness in $\hat{u}$-direction at least a, the Ψ-terms in the sum do not read the environments below level zero and consequently the sum is a function of $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots)$. Next one can check for $k \geq 1$ that

$$\Phi(\mathcal{S}_{k-1}, \mathcal{S}_k, \mathcal{S}_{k+1}, \dots) = \sum_{j=\tau_1(X_{\tau_{k-1}+\cdot} - X_{\tau_{k-1}})}^{\tau_2(X_{\tau_{k-1}+\cdot} - X_{\tau_{k-1}})-1} \Psi\big(T_{X_{\tau_{k-1}+j} - X_{\tau_{k-1}}}(T_{X_{\tau_{k-1}}}\omega)\big) = \varphi_k.$$

Now the sum of φ-terms in (4.8) can be decomposed into

$$\varphi_0 + \varphi_1 + \sum_{k=1}^{m-2} \Phi(\mathcal{S}_k, \mathcal{S}_{k+1}, \mathcal{S}_{k+2}, \dots).$$

The limit (4.8) follows because the slabs $(\mathcal{S}_k)_{k\geq 1}$ are i.i.d. and the finite initial terms $\varphi_0 + \varphi_1$ are eliminated by the m^{-1} factor.

Let $\alpha_n = \inf\{k : \tau_k \geq n\}$. Bounds (3.7)–(3.8) give finite moments of all orders to the increments $\tau_k - \tau_{k-1}$ and this implies that $n^{-1}(\tau_{\alpha_n-1} - \tau_{\alpha_n}) \to 0$ P_0-almost surely.

Consequently (4.8) yields the next limit, for another constant b:

$$\lim_{n\to\infty} n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega) = b \qquad P_0\text{-almost surely.} \qquad (4.9)$$

By boundedness this limit is valid also in $L^1(P_0)$ and the initial point of the walk is immaterial by shift-invariance of $\mathbb{P}$. Let $\ell > 0$ and choose a small $\varepsilon_0 > 0$. Abbreviate

$$G_{n,x}(\omega) = E_x^\omega\Big[\,\Big|n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega) - b\,\Big|\mathbb{1}\big\{\inf_{j\geq 0} X_j\cdot\hat{u} \geq X_0\cdot\hat{u} - \varepsilon_0\ell/2\big\}\Big].$$

Let

$$\mathcal{I} = \{x\in\mathbb{Z}^d : x\cdot\hat{u}\geq\varepsilon_0\ell, |x|\leq A\ell\}$$

for some constant A. Use the bound (4.1) on the variation distance and the fact that the functions $G_{n,x}(\omega)$ are uniformly bounded over all x,n,ω, and, if ℓ is large enough relative to a and ε_0, for $x\in\mathcal{I}$ the function $G_{n,x}$ is $\mathfrak{S}_{\varepsilon_0\ell/3}$-measurable.

$$\mathbb{P}_\infty\Big\{\sum_{x\in\mathcal{I}}P_0^\omega[X_\ell = x]G_{n,x}(\omega)\geq\varepsilon_1\Big\}\leq\sum_{x\in\mathcal{I}}\mathbb{P}_\infty\{G_{n,x}(\omega)\geq\varepsilon_1/(C\ell^d)\}$$

$$\leq C\ell^d\varepsilon_1^{-1}\sum_{x\in\mathcal{I}}\mathbb{E}_\infty G_{n,x}\leq C\ell^d\varepsilon_1^{-1}\sum_{x\in\mathcal{I}}\mathbb{E}G_{n,x} + C\ell^{2d}\varepsilon_1^{-1}e^{-c\varepsilon_0\ell/3}.$$

By (4.9) $\mathbb{E}G_{n,x}\to 0$ for any fixed x. Thus from above we get for any fixed ℓ,

$$\lim_{n\to\infty} E_0^\infty\big[\,\mathbb{1}\{X_\ell\in\mathcal{I}\}G_{n,X_\ell}\big]\leq\varepsilon_1 + C\ell^{2d}\varepsilon_1^{-1}e^{-c\varepsilon_0\ell/3}. \qquad (4.10)$$

The reader should bear in mind that the constant C is changing from line to line. Finally, we write

$$\overline{\lim_{n\to\infty}} E_0^\infty\Big|n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega) - b\,\Big|$$

$$\leq\overline{\lim_{n\to\infty}} E_0^\infty\Big[\mathbb{1}\{X_\ell\in\mathcal{I}\}\,\Big|n^{-1}\sum_{j=\ell}^{n+\ell-1}\Psi(T_{X_j}\omega) - b\,\Big|\mathbb{1}\big\{\inf_{j\geq\ell} X_j\cdot\hat{u}\geq X_\ell\cdot\hat{u} - \varepsilon_0\ell/2\big\}\Big]$$

$$+ CP_0^\infty\{X_\ell\notin\mathcal{I}\} + CP_0^\infty\big\{\inf_{j\geq\ell} X_j\cdot\hat{u} < X_\ell\cdot\hat{u} - \varepsilon_0\ell/2\big\}$$

$$\leq\overline{\lim_{n\to\infty}} E_0^\infty\big[\,\mathbb{1}\{X_\ell\in\mathcal{I}\}G_{n,X_\ell}\big] + CP_0^\infty\{X_\ell\cdot\hat{u} < \varepsilon_0\ell\}$$

$$+ CP_0^\infty\{\,|X_\ell| > A\ell\} + CE_0^\infty P_{X_\ell}^\omega\big\{\inf_{j\geq 0} X_j\cdot\hat{u} < X_0\cdot\hat{u} - \varepsilon_0\ell/2\big\}.$$

As pointed out, $\mathbb{P}_\infty$ satisfies Lemma 3.1 because hypotheses (N) and (M) were built into the space Ω_0 that supports $\mathbb{P}_\infty$. This enables us to make the error probabilities above small. Consequently, if we first pick ε_0 and ε_1 small enough, A large enough, then ℓ large, and apply (4.10), we will have shown (4.6). Ergodicity of $\mathbb{P}_\infty$ has been shown. This concludes the proof of Theorem 4.1.

Thereom 4.2 has also been established. It follows from the combination of (4.6) and (4.9). □

5. Change of measure

There are several stages in the proof where we need to check that a desired conclusion is not affected by choice between $\mathbb{P}$ and $\mathbb{P}_\infty$. We collect all instances of such transfers in this section. The standing assumptions of this section are that $\mathbb{P}$ is an i.i.d. product measure that satisfies Hypotheses (N) and (M), and that $\mathbb{P}_\infty$ is the measure given by Theorem 4.1. We show first that $\mathbb{P}_\infty$ can be replaced with $\mathbb{P}$ in the key condition (2.1) of Theorem 2.1.

LEMMA 5.1. *The velocity v defined by (2.5) satisfies $v = \mathbb{E}_\infty(D)$. There exists a constant C such that*

$$|E_0(X_n) - n\mathbb{E}_\infty(D)| \le C \qquad \text{for all } n \ge 1. \tag{5.1}$$

Proof. We start by showing $v = \mathbb{E}_\infty(D)$. The uniform exponential tail in the definition (4.3) of $\mathcal{P}_0$ makes the function $D(\omega)$ bounded and continuous on Ω_0. By the Cesàro definition of $\mathbb{P}_\infty$,

$$\mathbb{E}_\infty(D) = \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \mathbb{E}_k(D) = \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} E_0[D(T_{X_k}\omega)].$$

The moment bounds (3.7)–(3.9) imply that the law of large numbers $n^{-1}X_n \to v$ holds also in $L^1(P_0)$. From this and the Markov property

$$v = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_0(X_{k+1} - X_k) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_0[D(T_{X_k}\omega)].$$

We have proved $v = \mathbb{E}_\infty(D)$.

The variables $(X_{\tau_{j+1}} - X_{\tau_j}, \tau_{j+1} - \tau_j)_{j \ge 1}$ are i.i.d. with sufficient moments by (3.7)–(3.9). With $\alpha_n = \inf\{j \ge 1 : \tau_j - \tau_1 \ge n\}$ Wald's identity gives

$$E_0(X_{\tau_{\alpha_n}} - X_{\tau_1}) = E_0(\alpha_n)E_0(X_{\tau_1}|\beta = \infty) \quad \text{and} \quad E_0(\tau_{\alpha_n} - \tau_1) = E_0(\alpha_n)E_0(\tau_1|\beta = \infty).$$

Consequently, by the definition (2.5) of v,

$$E_0(X_n) - nv = vE_0(\tau_{\alpha_n} - \tau_1 - n) - E_0(X_{\tau_{\alpha_n}} - X_{\tau_1} - X_n).$$

It remains to show that $E_0(\tau_{\alpha_n} - \tau_1 - n)$ and $E_0(X_{\tau_{\alpha_n}} - X_{\tau_1} - X_n)$ are bounded by constants. We do this with a simple renewal argument. Let $Y_j = \tau_{j+1} - \tau_j$ for $j \ge 1$ and $V_0 = 0$, $V_m = Y_1 + \cdots + Y_m$. The quantity to bound is the forward recurrence time $B_n = \min\{k \ge 0 : n + k \in \{V_m\}\}$ because $\tau_{\alpha_n} - \tau_1 - n = B_n$.

We can write

$$B_n = (Y_1 - n)^+ + \sum_{k=1}^{n-1} \mathbb{I}\{Y_1 = k\}B_{n-k} \circ \theta$$

where θ shifts the sequence $\{Y_k\}$ and makes $B_{n-k} \circ \theta$ independent of Y_1. The two main terms on the right multiply to zero, so for any integer $p \geq 1$

$$B_n^p = ((Y_1 - n)^+)^p + \sum_{k=1}^{n-1} \mathbb{I}\{Y_1 = k\}(B_{n-k} \circ \theta)^p.$$

Set $z(n) = E_0((Y_1 - n)^+)^p$. Moment bounds (3.7)–(3.8) give $E_0(Y_1^{p+1}) < \infty$ which implies $\sum z(n) < \infty$. Taking expectations and using independence gives the discrete renewal equation

$$E_0 B_n^p = z(n) + \sum_{k=1}^{n-1} P_0(Y_1 = k) E_0 B_{n-k}^p.$$

Induction on n shows that $E_0 B_n^p \leq \sum_{k=1}^{n} z(k) \leq C(p)$ for all n. In particular, $E_0(\tau_{\alpha_n} - \tau_1 - n)^p$ is bounded by a constant uniformly over n. To extend this to $E_0 |X_{\tau_{\alpha_n}} - X_{\tau_1} - X_n|^p$ apply an argument like the one given for (3.9) at the end of Section 3. $\qquad\square$

PROPOSITION 5.2. *Assume that there exists an $\bar{\alpha} < 1/2$ such that*

$$\mathbb{E}\left(|E_0^\omega(X_n) - E_0(X_n)|^2\right) = O(n^{2\bar{\alpha}}). \tag{5.2}$$

Then condition (2.1) *is satisfied for some $\alpha < 1/2$.*

Proof. By (5.1) assumption (5.2) turns into

$$\mathbb{E}\left(|E_0^\omega(X_n) - nv|^2\right) = O(n^{2\bar{\alpha}}). \tag{5.3}$$

In the rest of this proof we use the conclusions of Lemma 3.1 under $\mathbb{P}_\infty$ instead of $\mathbb{P}$. This is justified by part (c) of Theorem 4.1.

For $k \geq 1$, recall that $\lambda_k = \inf\{n \geq 0 : (X_n - X_0) \cdot \hat{u} \geq k\}$. Take $k = [n^\rho]$ for a small enough $\rho > 0$. The point of the proof is to let the walk run up to a high level k so that expectations under $\mathbb{P}_\infty$ can be profitably related to expectations under $\mathbb{P}$ through the variation distance bound (4.1). Estimation is needed to remove the dependence on the environment on low levels. First compute as follows.

$$\mathbb{E}_\infty\left[|E_0^\omega(X_n - nv)|^2\right] = \mathbb{E}_\infty\left[|E_0^\omega(X_n - nv, \lambda_k \leq n) + E_0^\omega(X_n - nv, \lambda_k > n)|^2\right]$$

$$\leq 2\mathbb{E}_\infty\left[\left|E_0^\omega(X_n - X_{\lambda_k} - (n - \lambda_k)v, \lambda_k \leq n) - E_0^\omega(\lambda_k v, \lambda_k \leq n) + E_0^\omega(X_{\lambda_k}, \lambda_k \leq n)^2\right|\right]$$

$$+ O\left(n^2 \mathbb{E}_\infty[P_0^\omega(\lambda_k > n)]\right)$$

$$\leq 8\mathbb{E}_\infty\left[\left|\sum_{\substack{0 \leq m \leq n \\ x \cdot \hat{u} \geq k}} P_0^\omega(X_m = x, \lambda_k = m) E_x^\omega\{X_{n-m} - x - (n - m)v\}\right|^2\right]$$

$$+ O(k^2 + n^2 e^{sk}(1 - s\delta/2)^n). \tag{5.4}$$

The last error term above is $O(n^{2\rho})$. We used the Cauchy-Schwarz inequality and Hypothesis (M) to get the second term in the first inequality, and then (3.1), (3.4), and (3.5) in the last inequality.

To handle the expectation on line (5.4) we introduce a spanning set of vectors that satisfy the main assumptions that $\hat{u}$ does. Namely, let $\{\hat{u}_i\}_{i=1}^d$ span $\mathbb{R}^d$ and satisfy these conditions: $|\hat{u} - \hat{u}_i| \le \delta/(2M)$, where δ and M are the constants from Hypotheses (N) and (M), and

$$\hat{u} = \sum_{i=1}^d \alpha_i \hat{u}_i \quad \text{with } \alpha_i > 0. \tag{5.5}$$

Then non-nestling (N) holds for each $\hat{u}_i$ with constant $\delta/2$, and all the conclusions of Lemma 3.1 hold when $\hat{u}$ is replaced by $\hat{u}_i$ and δ by $\delta/2$. Define the event $A_k = \{\inf_i X_i \cdot \hat{u} \ge k\}$ and the set

$$\Lambda = \{x \in \mathbb{Z}^d : \min_i x \cdot \hat{u}_i \ge 1\}.$$

The point of introducing Λ is that the number of points x in Λ on level $x \cdot \hat{u} = \ell > 0$ is of order $O(\ell^{d-1})$.

By Jensen's inequality the expectation on line (5.4) is bounded by

$$2\mathbb{E}_\infty\left[\sum_{\substack{x\in\Lambda,\, x\cdot\hat{u}\ge k \\ 0\le m\le n}} P_0^\omega(X_m = x, \lambda_k = m)\big|E_x^\omega\{X_{n-m} - x - (n-m)v, A_{k/2}\}\big|^2\right] \tag{5.6}$$

$$+ \mathbb{E}_\infty\left[\sum_{\substack{0\le m\le n \\ x\notin\Lambda}} P_0^\omega(X_m = x, \lambda_k = m)\big|E_x^\omega\{X_{n-m} - x - (n-m)v\}\big|^2\right]$$

$$+ 2\mathbb{E}_\infty\left[\sum_{\substack{0\le m\le n \\ x\cdot\hat{u}\ge k}} P_0^\omega(X_m = x, \lambda_k = m)\big|E_x^\omega\{X_{n-m} - x - (n-m)v, A_{k/2}^c\}\big|^2\right].$$

By Cauchy-Schwarz, Hypothesis (M) and (3.1), the third term is $O(n^2 e^{-sk/2}) = O(1)$. The second term is of order

$$n^2 \max_i \mathbb{E}_\infty[P_0^\omega(X_{\lambda_k}\cdot\hat{u}_i < 1)] \le n^2\max_i\sum_{m\ge 1}\mathbb{E}_\infty\left[\left(P_0^\omega(X_m\cdot\hat{u}_i < 1)P_0^\omega(X_m\cdot\hat{u}\ge k)\right)^{1/2}\right]$$

$$\le e^{s/2}n^2\sum_{m\ge 1}(1 - s\delta/4)^{m/2}e^{-\mu k/2}e^{\mu M|\hat{u}|m/2}$$

$$= O(n^2 e^{-\mu k/2}) = O(1),$$

for μ small enough. It remains to bound the term on line (5.6). To this end, by Cauchy-Schwarz, (3.2) and (3.1),

$$P_0^\omega(X_m = x, \lambda_k = m) \le \{e^{-sx\cdot\hat{u}/2}e^{sM|\hat{u}|m/2}\wedge 1\}\times\{e^{\mu k/2}(1-\mu\delta/2)^{(m-1)/2}\wedge 1\} \equiv p_{x,m,k}.$$

Notice that

$$\sum_{x\in\Lambda}\{e^{-sx\cdot\hat{u}/2}e^{sM|\hat{u}|m/2}\wedge 1\} = O(m^d)$$

and

$$\sum_{m\geq 1} m^d \{e^{\mu k/2}(1-\mu\delta/2)^{(m-1)/2}\wedge 1\} = O(k^{d+1}).$$

Substitute these back into line (5.6) to eliminate the quenched probability coefficients. The quenched expectation in (5.6) is $\mathfrak{S}_{k/2}$-measurable. Consequently variation distance bound (4.1) allows us to switch back to $\mathbb{P}$ and get this upper bound for line (5.6):

$$2\sum_{\substack{x\in\Lambda,\,x\cdot\hat{u}\geq k \\ 0\leq m\leq n}} p_{x,m,k}\mathbb{E}[|E_x^\omega\{X_{n-m}-x-(n-m)v, A_{k/2}\}|^2] + O(k^{d+1}n^2 e^{-ck/2}).$$

The error term is again $O(1)$.

Now insert $A_{k/2}^c$ back inside the quenched expectation, incurring another error term of order $O(k^{d+1}n^2 e^{-sk/2}) = O(1)$. Using the shift-invariance of $\mathbb{P}$, along with (5.3), and collecting all of the above error terms, we get

$$\mathbb{E}_\infty\left[\,|E_0^\omega(X_n - nv)|^2\,\right]$$
$$= \sum_{\substack{x\in\Lambda,\,x\cdot\hat{u}\geq k \\ 0\leq m\leq n}} p_{x,m,k}\mathbb{E}[\,|E_x^\omega\{X_{n-m}-x-(n-m)v\}|^2] + O(n^{2\rho})$$
$$= O(k^{d+1}n^{2\bar\alpha} + n^{2\rho}) = O(n^{\rho(d+1)+2\bar\alpha}).$$

Pick $\rho > 0$ small enough so that $2\alpha = \rho(d+1) + 2\bar\alpha < 1$. The conclusion (2.1) follows. $\qquad\square$

Once we have verified the assumptions of Theorem 2.1 we have the CLT under $\mathbb{P}_\infty$-almost every ω. But we want the CLT under $\mathbb{P}$-almost every ω. Thus as the final point of this section we prove the transfer of the central limit theorem from $\mathbb{P}_\infty$ to $\mathbb{P}$. This is where we use the ergodic theorem, Theorem 4.2. Let W be the probability distribution of the Brownian motion with diffusion matrix $\mathfrak{D}$.

LEMMA 5.3. *Suppose the weak convergence $Q_n^\omega \Rightarrow W$ holds for $\mathbb{P}_\infty$-almost every ω. Then the same is true for $\mathbb{P}$-almost every ω.*

Proof. It suffices to show that for any bounded uniformly continuous F on $D_{\mathbb{R}^d}[0,\infty)$ and any $\delta > 0$

$$\varlimsup_{n\to\infty} E_0^\omega[F(B_n)] \leq \int F\,dW + \delta \quad \mathbb{P}\text{-a.s.}$$

By considering also $-F$ this gives $E_0^\omega[F(B_n)] \to \int F\,dW$ P_0-a.s. for each such function. A countable collection of them determines weak convergence.

Fix such an F. Let $c = \int F\,dW$ and

$$\overline{h}(\omega) = \varlimsup_{n\to\infty} E_0^\omega[F(B_n)].$$

For $\ell > 0$ define the events

$$A_{-\ell} = \{\inf_{n \geq 0} X_n \cdot \hat{u} \geq -\ell\}$$

and then

$$\overline{h}_\ell(\omega) = \varlimsup_{n \to \infty} E_0^\omega[F(B_n), A_{-\ell}] \quad \text{and} \quad \Psi(\omega) = \mathbb{1}\{\omega : \overline{h}_\ell(\omega) \leq c + \tfrac{1}{2}\delta\}.$$

The assumed quenched CLT under $\mathbb{P}_\infty$ gives $\mathbb{P}_\infty\{\overline{h} = c\} = 1$. By (3.1), and by its extension to $\mathbb{P}_\infty$ in Theorem 4.1(c), there are constants $0 < C, s < \infty$ such that

$$|\overline{h}(\omega) - \overline{h}_\ell(\omega)| \leq C e^{-s\ell}$$

uniformly over all ω that support both $\mathbb{P}$ and $\mathbb{P}_\infty$. Consequently if $\delta > 0$ is given, $\mathbb{E}_\infty \Psi = 1$ for large enough ℓ. Since Ψ is $\mathfrak{S}_{-\ell}$-measurable Theorem 4.2 implies that

$$n^{-1} \sum_{j=1}^{n} \Psi(T_{X_j}\omega) \to 1 \quad P_0\text{-a.s.}$$

By increasing ℓ if necessary we can ensure that $\{\overline{h}_\ell \leq c + \tfrac{1}{2}\delta\} \subset \{\overline{h} \leq c + \delta\}$ and conclude that the stopping time

$$\zeta = \inf\{n \geq 0 : \overline{h}(T_{X_n}\omega) \leq c + \delta\}$$

is P_0-a.s. finite. From the definitions we now have

$$\varlimsup_{n \to \infty} E_0^{T_{X_\zeta}\omega}[F(B_n)] \leq \int F\, dW + \delta \quad P_0\text{-a.s.}$$

Then by bounded convergence

$$\varlimsup_{n \to \infty} E_0^\omega E_0^{T_{X_\zeta}\omega}[F(B_n)] \leq \int F\, dW + \delta \quad \mathbb{P}\text{-a.s.}$$

Since ζ is a finite stopping time, the strong Markov property, the uniform continuity of F and the exponential moment bound (3.2) on X-increments imply

$$\varlimsup_{n \to \infty} E_0^\omega[F(B_n)] \leq \int F\, dW + \delta \quad \mathbb{P}\text{-a.s.}$$

This concludes the proof. $\qquad\square$

6. Reduction to path intersections

The preceding sections have reduced the proof of the main result Theorem 1.1 to proving the estimate

$$\mathbb{E}\left(|E_0^\omega(X_n) - E_0(X_n)|^2\right) = O(n^{2\alpha}) \quad \text{for some } \alpha < 1/2. \tag{6.1}$$

The next reduction takes us to the expected number of intersections of the paths of two independent walks X and $\widetilde{X}$ in the same environment. The argument uses a decomposition into martingale differences through an ordering of lattice sites. This idea

for bounding a variance is natural and has been used in RWRE earlier by Bolthausen and Sznitman [2].

Let $P_{0,0}^\omega$ be the quenched law of the walks $(X, \widetilde{X})$ started at $(X_0, \widetilde{X}_0) = (0,0)$ and $P_{0,0} = \int P_{0,0}^\omega \mathbb{P}(d\omega)$ the averaged law with expectation operator $E_{0,0}$. The set of sites visited by a walk is denoted by $X_{[0,n)} = \{X_k : 0 \le k < n\}$ and $|A|$ is the number of elements in a discrete set A.

PROPOSITION 6.1. *Let $\mathbb{P}$ be an i.i.d. product measure and satisfy Hypotheses* (N) *and* (M). *Assume that there exists an $\bar{\alpha} < 1/2$ such that*

$$E_{0,0}(|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|) = O(n^{2\bar{\alpha}}). \tag{6.2}$$

Then condition (6.1) *is satisfied.*

Proof. For $L \ge 0$, define $\mathcal{B}(L) = \{x \in \mathbb{Z}^d : |x| \le L\}$. Fix $n \ge 1$, $c > |\hat{u}|$, and let $(x_j)_{j \ge 1}$ be some fixed ordering of $\mathcal{B}(cMn)$ satisfying

$$\forall i \ge j : x_i \cdot \hat{u} \ge x_j \cdot \hat{u}.$$

For $B \subset \mathbb{Z}^d$ let $\mathfrak{S}_B = \sigma\{\omega_x : x \in B\}$. Let $A_j = \{x_1, \ldots, x_j\}$, $\zeta_0 = E_0(X_n)$, and for $j \ge 1$

$$\zeta_j = \mathbb{E}(E_0^\omega(X_n)|\mathfrak{S}_{A_j}).$$

$(\zeta_j - \zeta_{j-1})_{j \ge 1}$ is a sequence of $L^2(\mathbb{P})$-martingale differences and we have

$$\mathbb{E}[\,|E_0^\omega(X_n) - E_0(X_n)|^2\,] \tag{6.3}$$
$$\le 2\mathbb{E}\big[\,|E_0(X_n) - \mathbb{E}\{E_0^\omega(X_n)|\mathfrak{S}_{\mathcal{B}(cMn)}\}|^2\,\big]$$
$$+ 2\mathbb{E}\big[\,\big|E_0^\omega(X_n, \max_{i \le n}|X_i| > cMn)$$
$$- \mathbb{E}\{E_0^\omega(X_n, \max_{i \le n}|X_i| > cMn)\,|\,\mathfrak{S}_{\mathcal{B}(cMn)}\}\big|^2\,\big]$$
$$\le 2 \sum_{j=1}^{|\mathcal{B}(cMn)|} \mathbb{E}(\,|\zeta_j - \zeta_{j-1}|^2\,) + O(n^3 e^{-sM(c-|\hat{u}|)n}). \tag{6.4}$$

In the last inequality we have used (3.2). The error is $O(1)$. For $z \in \mathbb{Z}^d$ define half-spaces

$$\mathcal{H}(z) = \{x \in \mathbb{Z}^d : x \cdot \hat{u} > z \cdot \hat{u}\}.$$

Since $A_{j-1} \subset A_j \subset \mathcal{H}(x_j)^c$,

$$\mathbb{E}(|\zeta_j - \zeta_{j-1}|^2)$$
$$= \int \mathbb{P}(d\omega_{A_j})\Big|\iint \mathbb{P}(d\omega_{A_j^c})\mathbb{P}(d\tilde{\omega}_{x_j})\big(E_0^\omega(X_n) - E_0^{\langle\omega,\tilde{\omega}_{x_j}\rangle}(X_n)\big)\Big|^2$$
$$\le \iint \mathbb{P}(d\omega_{\mathcal{H}(x_j)^c})\mathbb{P}(d\tilde{\omega}_{x_j})\Big|\int \mathbb{P}(d\omega_{\mathcal{H}(x_j)})\big(E_0^\omega(X_n) - E_0^{\langle\omega,\tilde{\omega}_{x_j}\rangle}(X_n)\big)\Big|^2. \tag{6.5}$$

Above $\langle\omega,\tilde{\omega}_{x_j}\rangle$ denotes an environment obtained from ω by replacing ω_{x_j} with $\tilde{\omega}_{x_j}$.

We fix a point $z = x_j$ to develop a bound for the expression above, and then return to collect the estimates. Abbreviate $\tilde{\omega} = \langle \omega, \tilde{\omega}_{x_j} \rangle$. Consider two walks X_n and $\widetilde{X}_n$ starting at 0. X_n obeys environment ω, while $\widetilde{X}_n$ obeys $\tilde{\omega}$. We can couple the two walks so that they stay together until the first time they visit z. Until a visit to z happens, the walks are identical. So we write

$$\int \mathbb{P}(d\omega_{\mathcal{H}(z)}) \big(E_0^{\omega}(X_n) - E_0^{\tilde{\omega}}(X_n) \big) \tag{6.6}$$

$$= \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) \sum_{m=0}^{n-1} P_0^{\omega}(H_z = m) \big(E_z^{\omega}(X_{n-m} - z) - E_z^{\tilde{\omega}}(X_{n-m} - z) \big)$$

$$= \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) \sum_{m=0}^{n-1} \sum_{\ell > 0} P_0^{\omega}(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell) \tag{6.7}$$

$$\times \big(E_z^{\omega}(X_{n-m} - z) - E_z^{\tilde{\omega}}(X_{n-m} - z) \big).$$

Decompose $\mathcal{H}(z) = \mathcal{H}_\ell(z) \cup \mathcal{H}'_\ell(z)$ where

$$\mathcal{H}_\ell(z) = \{ x \in \mathbb{Z}^d : z \cdot \hat{u} < x \cdot \hat{u} < z \cdot \hat{u} + \ell \} \text{ and } \mathcal{H}'_\ell(z) = \{ x \in \mathbb{Z}^d : x \cdot \hat{u} \ge z \cdot \hat{u} + \ell \}.$$

Take a single (ℓ, m) term from the sum in (6.7) and only the expectation $E_z^{\omega}(X_{n-m} - z)$.

$$\int \mathbb{P}(d\omega_{\mathcal{H}(z)}) P_0^{\omega}(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell)$$

$$\times E_z^{\omega}(X_{n-m} - z)$$

$$= \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) P_0^{\omega}(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell) \tag{6.8}$$

$$\times E_z^{\omega}(X_{\tau_1^{(\ell)} + n - m} - X_{\tau_1^{(\ell)}})$$

$$+ \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) P_0^{\omega}(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell) \tag{6.9}$$

$$\times E_z^{\omega}(X_{n-m} - X_{\tau_1^{(\ell)} + n - m} + X_{\tau_1^{(\ell)}} - z)$$

The parameter ℓ in the regeneration time $\tau_1^{(\ell)}$ of the walk started at z ensures that the subsequent walk $X_{\tau_1^{(\ell)} +}$. stays in $\mathcal{H}'_\ell(z)$. Below we make use of this to get independence from the environments in $\mathcal{H}'_\ell(z)^c$. By (3.9) the quenched expectation in (6.9) can be bounded by $C_p \ell^p$, for any $p > 1$.

Integral (6.8) is developed further as follows.

$$
\int \mathbb{P}(d\omega_{\mathcal{H}(z)}) P_0^\omega(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell)
$$
$$
\times E_z^\omega(X_{\tau_1^{(\ell)}+n-m} - X_{\tau_1^{(\ell)}})
$$
$$
= \int \mathbb{P}(d\omega_{\mathcal{H}_\ell(z)}) P_0^\omega(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell)
$$
$$
\times \int \mathbb{P}(d\omega_{\mathcal{H}'_\ell(z)}) E_z^\omega(X_{\tau_1^{(\ell)}+n-m} - X_{\tau_1^{(\ell)}})
$$
$$
= \int \mathbb{P}(d\omega_{\mathcal{H}_\ell(z)}) P_0^\omega(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell)
$$
$$
\times E_z(X_{\tau_1^{(\ell)}+n-m} - X_{\tau_1^{(\ell)}} | \mathfrak{S}_{\mathcal{H}'_\ell(z)^c})
$$
$$
= \int \mathbb{P}(d\omega_{\mathcal{H}_\ell(z)}) P_0^\omega(H_z = m, \ell - 1 \le \max_{0 \le j \le m} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell) \tag{6.10}
$$
$$
\times E_0(X_{n-m} | \beta = \infty).
$$

The last equality above comes from the regeneration structure, see Proposition 1.3 in Sznitman-Zerner [17]. The σ-algebra $\mathfrak{S}_{\mathcal{H}'_\ell(z)^c}$ is contained in the σ-algebra $\mathcal{G}_1$ defined by (1.22) of [17] for the walk starting at z.

The last quantity (6.10) above reads the environment only until the first visit to z, hence does not see the distinction between ω and $\tilde{\omega}$. Hence when the integral (6.7) is developed separately for ω and $\tilde{\omega}$ into the sum of integrals (6.8) and (6.9), integrals (6.8) for ω and $\tilde{\omega}$ cancel each other. We are left only with two instances of integral (6.9), one for both ω and $\tilde{\omega}$. The last quenched expectation in (6.9) we bound by $C_p \ell^p$ as was mentioned above.

Going back to (6.6), we get this bound:

$$
\left| \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) \left(E_0^\omega(X_n) - E_0^{\tilde{\omega}}(X_n) \right) \right|
$$
$$
\le C_p \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) \sum_{\ell > 0} \ell^p P_0^\omega(H_z < n, \ell - 1 \le \max_{0 \le j \le H_z} X_j \cdot \hat{u} - z \cdot \hat{u} < \ell)
$$
$$
\le C_p \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) E_0^\omega[(M_{H_z} - z \cdot \hat{u})^p \mathbb{I}\{H_z < n\}]
$$
$$
\le C_p n^{p\varepsilon} \int \mathbb{P}(d\omega_{\mathcal{H}(z)}) P_0^\omega(H_z < n) + C_p s^{-p} e^{-sn^\varepsilon/2}.
$$

For the last inequality we used (3.6) with $\ell = n^\varepsilon$ and some small $\varepsilon, s > 0$. Square, take $z = x_j$, integrate as in (6.5), and use Jensen's inequality to bring the square inside the integral to get

$$
\mathbb{E}(|\zeta_j - \zeta_{j-1}|^2) \le 2C_p n^{2p\varepsilon} \mathbb{E}[|P_0^\omega(H_{x_j} < n)|^2] + 2C_p s^{-2p} e^{-sn^\varepsilon}.
$$

Substitute these bounds into line (6.4) and note that the error there is $O(1)$.

$$\mathbb{E}\big[\,|E_0^\omega(X_n) - E_0(X_n)|^2\,\big]$$
$$\leq C_p n^{2p\varepsilon} \sum_z \mathbb{E}\big[\,|P_0^\omega(H_z < n)|^2\,\big] + O(n^d s^{-2p} e^{-sn^\varepsilon}) + O(1)$$
$$= C_p n^{2p\varepsilon} \sum_z P_{0,0}(z \in X_{[0,n)} \cap \tilde{X}_{[0,n)}) + O(1)$$
$$= C_p n^{2p\varepsilon} E_{0,0}\big[\,|X_{[0,n)} \cap \tilde{X}_{[0,n)}|\,\big] + O(1).$$

Utilize assumption (6.2) and take $\varepsilon > 0$ small enough so that $2\alpha = 2p\varepsilon + 2\bar{\alpha} < 1$. (6.1) has been verified. $\qquad\square$

7. Bound on intersections

The remaining piece of the proof of Theorem 1.1 is this estimate:

$$E_{0,0}(\,|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|\,) = O(n^{2\alpha}) \quad \text{for some } \alpha < 1/2. \tag{7.1}$$

X and $\widetilde{X}$ are two independent walks in a common environment with quenched distribution $P_{x,y}^\omega[X_{0,\infty} \in A, \widetilde{X}_{0,\infty} \in B] = P_x^\omega(A) P_y^\omega(B)$ and averaged distribution $E_{x,y}(\cdot) = \mathbb{E}P_{x,y}^\omega(\cdot)$.

To deduce the sublinear bound we introduce regeneration times at which both walks regenerate on the same level in space (but not necessarily at the same time). Intersections happen only within the regeneration slabs, and the expected number of intersections decays exponentially in the distance between the points of entry of the walks in the slab. From regeneration to regeneration the difference of the two walks operates like a Markov chain. This Markov chain can be approximated by a symmetric random walk. Via this preliminary work the required estimate boils down to deriving a Green function bound for a Markov chain that can be suitably approximated by a symmetric random walk. This part is relegated to an appendix. Except for the appendix, we complete the proof of the functional central limit theorem in this section.

To aid our discussion of a pair of walks $(X, \widetilde{X})$ we introduce some new notation. We write $\theta^{m,n}$ for the shift on pairs of paths: $\theta^{m,n}(x_{0,\infty}, y_{0,\infty}) = (\theta^m x_{0,\infty}, \theta^n y_{0,\infty})$. If we write separate expectations for X and $\widetilde{X}$ under $P_{x,y}^\omega$, these are denoted by E_x^ω and $\widetilde{E}_y^\omega$.

By a *joint stopping time* we mean pair $(\alpha, \tilde{\alpha})$ that satisfies $\{\alpha = m, \tilde{\alpha} = n\} \in \sigma\{X_{0,m}, \widetilde{X}_{0,n}\}$. Under the distribution $P_{x,y}^\omega$ the walks X and $\widetilde{X}$ are independent. Consequently if $\alpha \vee \tilde{\alpha} < \infty$ $P_{x,y}^\omega$-almost surely then for any events A and B,

$$P_{x,y}^\omega[(X_{0,\alpha}, \widetilde{X}_{0,\tilde{\alpha}}) \in A, \ (X_{\alpha,\infty}, \widetilde{X}_{\tilde{\alpha},\infty}) \in B]$$
$$= E_{x,y}^\omega\big[\mathbb{1}\{(X_{0,\alpha}, \widetilde{X}_{0,\tilde{\alpha}}) \in A\} P_{X_\alpha, \widetilde{X}_{\tilde{\alpha}}}^\omega \{(X_{0,\infty}, \widetilde{X}_{0,\infty}) \in B\}\big].$$

This type of joint restarting will be used without comment in the sequel.

For this section it will be convenient to have level stopping times and running maxima that are not defined relative to the initial level.

$$\gamma_\ell = \inf\{n \geq 0 : X_n \cdot \hat{u} \geq \ell\} \quad \text{and} \quad \gamma_\ell^+ = \inf\{n \geq 0 : X_n \cdot \hat{u} > \ell\}.$$

Since $\hat{u} \in \mathbb{Z}^d$, γ_ℓ^+ is simply an abbreviation for $\gamma_{\ell+1}$. Let $M_n = \sup\{X_i \cdot \hat{u} : i \leq n\}$ be the running maximum. $\widetilde{M}_n$, $\tilde{\gamma}_\ell$ and $\tilde{\gamma}_\ell^+$ are the corresponding quantities for the $\widetilde{X}$ walk. The first backtracking time for the $\widetilde{X}$ walk is $\tilde{\beta} = \inf\{n \geq 1 : \widetilde{X}_n \cdot \hat{u} < \widetilde{X}_0 \cdot \hat{u}\}$.

Define

$$L = \inf\{\ell > (X_0 \cdot \hat{u}) \wedge (\widetilde{X}_0 \cdot \hat{u}) : X_{\gamma_\ell} \cdot \hat{u} = \widetilde{X}_{\tilde{\gamma}_\ell} \cdot \hat{u} = \ell\}$$

as the first fresh common level after at least one walk has exceeded its starting level. Set $L = \infty$ if there is no such common level. When the walks are on a common level, their difference will lie in the hyperplane

$$\mathbb{V}_d = \{z \in \mathbb{Z}^d : z \cdot \hat{u} = 0\}.$$

We start with exponential tail bounds on the time to reach the common level.

LEMMA 7.1. *There exist constants $0 < a_1, a_2, C < \infty$ such that, for all $x, y \in \mathbb{Z}^d$, $m \geq 0$ and $\mathbb{P}$-a.e. ω,*

$$P_{x,y}^\omega(\gamma_L \vee \tilde{\gamma}_L \geq m) \leq Ce^{a_1|y\cdot\hat{u}-x\cdot\hat{u}|-a_2 m}. \tag{7.2}$$

For the proof we need a bound on the overshoot.

LEMMA 7.2. *There exist constants $0 < C, s < \infty$ such that, for any level k, any $b \geq 1$, any $x \in \mathbb{Z}^d$ such that $x \cdot \hat{u} \leq k$, and $\mathbb{P}$-a.e. ω,*

$$P_x^\omega[X_{\gamma_k} \cdot \hat{u} \geq k + b] \leq Ce^{-sb}. \tag{7.3}$$

Proof. From (3.1) it follows that for a constant C, for any level ℓ, any $x \in \mathbb{Z}^d$, and $\mathbb{P}$-a.e. ω,

$$E_x^\omega[\text{number of visits to level } \ell] = \sum_{n=0}^\infty P_x^\omega[X_n \cdot \hat{u} = \ell] \leq C. \tag{7.4}$$

(This is certainly clear if $x \cdot \hat{u} = \ell$. Otherwise wait until the process first lands on level ℓ, if ever.)

From this and the exponential moment hypothesis we deduce the required bound on the overshoots: for any k, any $x \in \mathbb{Z}^d$ such that $x \cdot \hat{u} \leq k$, and $\mathbb{P}$-a.e. ω,

$$P_x^\omega[X_{\gamma_k} \cdot \hat{u} \geq k + b] = \sum_{n=0}^{\infty} \sum_{z \cdot \hat{u} < k} P_x^\omega[\gamma_k > n, \, X_n = z, \, X_{n+1} \cdot \hat{u} \geq k + b]$$

$$= \sum_{\ell > 0} \sum_{z \cdot \hat{u} = k - \ell} \sum_{n=0}^{\infty} P_x^\omega[\gamma_k > n, \, X_n = z] P_z^\omega[X_1 \cdot \hat{u} \geq k + b]$$

$$\leq \sum_{\ell > 0} \sum_{z \cdot \hat{u} = k - \ell} \sum_{n=0}^{\infty} P_x^\omega[X_n = z] C e^{-s(\ell + b)}$$

$$\leq C e^{-sb} \sum_{\ell > 0} e^{-s\ell} \leq C e^{-sb}. \qquad \qquad \square$$

Proof of Lemma 7.1. Consider first γ_L, and let us restrict ourselves to the case where the initial points x, y satisfy $x \cdot \hat{u} < y \cdot \hat{u}$.

Perform an iterative construction of stopping times η_i, $\tilde{\eta}_i$ and levels $\ell(i)$, $\tilde{\ell}(i)$. Let $\eta_0 = \tilde{\eta}_0 = 0$, $x_0 = x$ and $y_0 = y$. $\ell(0)$ and $\tilde{\ell}(0)$ need not be defined. Suppose that the construction has been done to stage $i - 1$ with $x_{i-1} = X_{\eta_{i-1}}$, $y_{i-1} = \widetilde{X}_{\tilde{\eta}_{i-1}}$, and $x_{i-1} \cdot \hat{u} < y_{i-1} \cdot \hat{u}$. Then set

$$\ell(i) = X_{\gamma(y_{i-1} \cdot \hat{u})} \cdot \hat{u}, \; \tilde{\ell}(i) = \widetilde{X}_{\tilde{\gamma}(\ell(i))} \cdot \hat{u}, \; \eta_i = \gamma(\tilde{\ell}(i)) \; \text{ and } \; \tilde{\eta}_i = \tilde{\gamma}(X_{\eta_i} \cdot \hat{u} + 1).$$

In words, starting at (x_{i-1}, y_{i-1}) with y_{i-1} above x_{i-1}, let X reach the level of y_{i-1} and let $\ell(i)$ be the level X lands on; let $\widetilde{X}$ reach the level $\ell(i)$ and let $\tilde{\ell}(i)$ be the level $\widetilde{X}$ lands on. Now let X try to establish a new common level at $\tilde{\ell}(i)$ with $\widetilde{X}$: in other words, follow X until the time η_i it reaches level $\tilde{\ell}(i)$ or above, and stop it there. Finally, reset the situation by letting $\widetilde{X}$ reach a level strictly above the level of X_{η_i}, and stop it there at time $\tilde{\eta}_i$. The starting locations for the next step are $x_i = X_{\eta_i}$, $y_i = \widetilde{X}_{\tilde{\eta}_i}$ that satisfy $x_i \cdot \hat{u} < y_i \cdot \hat{u}$.

We show that within each step of the iteration there is a uniform lower bound on the probability that a fresh common level was found. For this purpose we utilize assumption (1.1) in the weaker form

$$\mathbb{P}\{\omega : \, P_0^\omega(X_{\gamma_1} \cdot \hat{u} = 1) \geq \kappa\} = 1. \tag{7.5}$$

Pick b large enough so that the bound in (7.3) is < 1. For $z, w \in \mathbb{Z}^d$ such that $z \cdot \hat{u} \geq w \cdot \hat{u}$ define a function

$$\psi(z, w) = P_{z,w}^\omega[\, X_{\gamma_k} \cdot \hat{u} = k \text{ for each } k \in \{z \cdot \hat{u}, \ldots, z \cdot \hat{u} + b\},$$
$$\widetilde{X}_{\tilde{\gamma}(z \cdot \hat{u})} \cdot \hat{u} - z \cdot \hat{u} \leq b\,]$$
$$\geq \kappa^b (1 - C e^{-sb}) \equiv \kappa_2 > 0.$$

The uniform lower bound comes from the independence of the walks, from (7.3) and from iterating assumption (7.5). By the Markov property

$$P^\omega_{x_{i-1},y_{i-1}}[\,X_{\gamma(\tilde\ell(i))} \cdot \hat u = \tilde\ell(i)]$$
$$\geq P^\omega_{x_{i-1},y_{i-1}}[\,\widetilde X_{\tilde\gamma(\ell(i))} \cdot \hat u - \ell(i) \leq b, \ \ X_{\gamma_k} \cdot \hat u = k \text{ for each } k \in \{\ell(i),\dots,\ell(i)+b\}\,]$$
$$\geq E^\omega_{x_{i-1},y_{i-1}}\big[\psi(X_{\gamma(y_{i-1}\cdot\hat u)}, y_{i-1})\big] \geq \kappa_2.$$

The first iteration on which the attempt to create a common level at $\tilde\ell(i)$ succeeds is

$$I = \inf\{i \geq 1 : X_{\gamma_{\tilde\ell(i)}} \cdot \hat u = \tilde\ell(i)\}.$$

Then $\tilde\ell(I)$ is a new fresh common level and consequently $L \leq \tilde\ell(I)$. This gives the upper bound

$$\gamma_L \leq \gamma_{\tilde\ell(I)}.$$

We develop an exponential tail bound for $\gamma_{\tilde\ell(I)}$, still under the assumption $x\cdot\hat u < y\cdot\hat u$.

From the uniform bound above and the Markov property we get

$$P^\omega_{x,y}[I > i] \leq (1 - \kappa_2)^i.$$

Lemma 7.2 gives an exponential bound

$$P^\omega_{x,y}\big[(\widetilde X_{\tilde\eta_i} - \widetilde X_{\tilde\eta_{i-1}}) \cdot \hat u \geq b\big] \leq C e^{-sb} \tag{7.6}$$

because the distance $(\widetilde X_{\tilde\eta_i} - \widetilde X_{\tilde\eta_{i-1}}) \cdot \hat u$ is a sum of four overshoots:

$$\big(\widetilde X_{\tilde\eta_i} - \widetilde X_{\tilde\eta_{i-1}}\big) \cdot \hat u = \big(\widetilde X_{\tilde\gamma(X_{\eta_i}\cdot\hat u+1)} \cdot \hat u - X_{\eta_i} \cdot \hat u - 1\big) + 1 + \big(X_{\gamma(\tilde\ell(i))} \cdot \hat u - \tilde\ell(i)\big)$$
$$+ \big(\widetilde X_{\tilde\gamma(\ell(i))} \cdot \hat u - \ell(i)\big) + \big(X_{\gamma(\widetilde X_{\tilde\eta_{i-1}}\cdot\hat u)} \cdot \hat u - \widetilde X_{\tilde\eta_{i-1}} \cdot \hat u\big).$$

Next, from the exponential tail bound on $\big(\widetilde X_{\tilde\eta_i} - \widetilde X_{\tilde\eta_{i-1}}\big) \cdot \hat u$ and from

$$\tilde\ell(i) \leq \widetilde X_{\tilde\eta_i} \cdot \hat u = \sum_{j=1}^{i}\big(\widetilde X_{\tilde\eta_j} - \widetilde X_{\tilde\eta_{j-1}}\big) \cdot \hat u + y \cdot \hat u$$

we get the large deviation estimate

$$P^\omega_{x,y}[\tilde\ell(i) \geq bi + y \cdot \hat u] \leq e^{-sbi} \quad \text{for } i \geq 1 \text{ and } b \geq b_0,$$

for some constants $0 < s < \infty$ (small enough) and $0 < b_0 < \infty$ (large enough). Combine this with the bound above on I to write

$$P^\omega_{x,y}[\tilde\ell(I) \geq a] \leq P^\omega_{x,y}[I > i] + P^\omega_{x,y}[\tilde\ell(i) \geq a]$$
$$\leq e^{-si} + e^{sy\cdot\hat u - sa} \leq 2e^{sy\cdot\hat u - sa}$$

where we assume $a \geq 2b_0 + y \cdot \hat u$ and set the integer $i = \lfloor b_0^{-1}(a - y \cdot \hat u)\rfloor$. Recall that $0 < s < \infty$ is a constant whose value can change from line to line.

From (3.1) and an exponential Chebyshev

$$P^\omega_{x,y}[\gamma_k > m] \le P^\omega_x[X_m \cdot \hat{u} \le k] \le e^{sk - sx \cdot \hat{u} - h_1 m}$$

for all $x, y \in \mathbb{Z}^d$, $k \in \mathbb{Z}$ and $m \ge 0$. Above and in the remainder of this proof h_1, h_2 and h_3 are small positive constants. Finally we derive

$$P^\omega_{x,y}[\gamma_{\tilde{\ell}(I)} > m] \le P^\omega_{x,y}[\tilde{\ell}(I) \ge k + x \cdot \hat{u}] + P^\omega_{x,y}[\gamma_{k+x \cdot \hat{u}} > m]$$

$$\le 2 e^{s(y-x) \cdot \hat{u} - sk} + e^{sk - h_1 m} \le C e^{s(y-x) \cdot \hat{u} - h_2 m}.$$

To justify the inequalities above assume $m \ge 4sb_0/h_1 > 4s/h_1$ and pick k in the range

$$\frac{h_1 m}{2s} + (y - x) \cdot \hat{u} \le k \le \frac{3 h_1 m}{4s} + (y - x) \cdot \hat{u}.$$

To summarize, at this point we have

$$P^\omega_{x,y}[\gamma_L > m] \le C e^{s(y-x) \cdot \hat{u} - h_2 m} \quad \text{for } x \cdot \hat{u} < y \cdot \hat{u}. \tag{7.7}$$

To extend this estimate to the case $x \cdot \hat{u} \ge y \cdot \hat{u}$, simply allow $\widetilde{X}$ to go above x and then apply (7.7). By an application of the overshoot bound (7.3) and (7.7) at the point $(x, \widetilde{X}_{\tilde{\gamma}(x \cdot \hat{u}+1)})$

$$P^\omega_{x,y}[\gamma_L > m] \le E^\omega_{x,y} P^\omega_{x, \widetilde{X}_{\tilde{\gamma}(x \cdot \hat{u}+1)}}[\gamma_L > m]$$

$$\le P^\omega_{x,y}[\widetilde{X}_{\tilde{\gamma}(x \cdot \hat{u}+1)} > x \cdot \hat{u} + \varepsilon m] + C e^{s\varepsilon m - h_2 m} \le C e^{-h_3 m}$$

if we take $\varepsilon > 0$ small enough.

We have proved the lemma for γ_L, and the same argument works for $\tilde{\gamma}_L$. □

Assuming that $X_0 \cdot \hat{u} = \widetilde{X}_0 \cdot \hat{u}$ define the joint stopping times

$$(\rho, \tilde{\rho}) = (\gamma^+_{M_{\beta \wedge \tilde{\beta}} \vee \widetilde{M}_{\beta \wedge \tilde{\beta}}}, \tilde{\gamma}^+_{M_{\beta \wedge \tilde{\beta}} \vee \widetilde{M}_{\beta \wedge \tilde{\beta}}})$$

and

$$(\nu_1, \tilde{\nu}_1) = \begin{cases} (\rho, \tilde{\rho}) + (\gamma_L, \tilde{\gamma}_L) \circ \theta^{\rho, \tilde{\rho}} & \text{if } \rho \vee \tilde{\rho} < \infty \\ \infty & \text{if } \rho = \tilde{\rho} = \infty. \end{cases} \tag{7.8}$$

Notice that ρ and $\tilde{\rho}$ are finite or infinite together, and they are infinite iff neither walk backtracks below its initial level ($\beta = \tilde{\beta} = \infty$). Let $\nu_0 = \tilde{\nu}_0 = 0$ and for $k \ge 0$ define

$$(\nu_{k+1}, \tilde{\nu}_{k+1}) = (\nu_k, \tilde{\nu}_k) + (\nu_1, \tilde{\nu}_1) \circ \theta^{\nu_k, \tilde{\nu}_k}.$$

Finally let $(\nu, \tilde{\nu}) = (\gamma_L, \tilde{\gamma}_L)$, $K = \sup\{k \ge 0 : \nu_k \vee \tilde{\nu}_k < \infty\}$, and

$$(\mu_1, \tilde{\mu}_1) = (\nu, \tilde{\nu}) + (\nu_K, \tilde{\nu}_K) \circ \theta^{\nu, \tilde{\nu}}. \tag{7.9}$$

These represent the first *common regeneration times* of the two paths. Namely, $X_{\mu_1} \cdot \hat{u} = \widetilde{X}_{\tilde{\mu}_1} \cdot \hat{u}$ and for all $n \ge 1$,

$$X_{\mu_1 - n} \cdot \hat{u} < X_{\mu_1} \cdot \hat{u} \le X_{\mu_1 + n} \cdot \hat{u} \text{ and } \widetilde{X}_{\tilde{\mu}_1 - n} \cdot \hat{u} < \widetilde{X}_{\tilde{\mu}_1} \cdot \hat{u} \le \widetilde{X}_{\tilde{\mu}_1 + n} \cdot \hat{u}.$$

Next we extend the exponential tail bound to the regeneration times.

LEMMA 7.3. *There exist constants $0 < C < \infty$ and $\bar{\eta} \in (0,1)$ such that, for all $x, y \in \mathbb{V}_d = \{z \in \mathbb{Z}^d : z \cdot \hat{u} = 0\}$, $k \geq 0$, and $\mathbb{P}$-a.e. ω, we have*

$$P_{x,y}^{\omega}(\mu_1 \vee \tilde{\mu}_1 \geq k) \leq C(1 - \bar{\eta})^k. \tag{7.10}$$

Proof. We prove geometric tail bounds successively for γ_1^+, γ_ℓ^+, $\gamma_{M_r}^+$, ρ, ν_1, ν_k, and finally for μ_1. To begin, (3.1) implies that

$$P_0^{\omega}(\gamma_1^+ \geq n) \leq P_0^{\omega}(X_{n-1} \cdot \hat{u} \leq 1) \leq e^{s_2}(1 - \eta_1)^{n-1}$$

with $\eta_1 = s_2 \delta/2$, for some small $s_2 > 0$. By summation by parts

$$E_0^{\omega}(e^{s_3 \gamma_1^+}) \leq e^{s_2} J_{s_3},$$

for a small enough $s_3 > 0$ and $J_s = 1 + (e^s - 1)/(1 - (1 - \eta_1)e^s)$. By the Markov property for $\ell \geq 1$,

$$E_0^{\omega}(e^{s_3 \gamma_\ell^+}) \leq \sum_{x \cdot \hat{u} > \ell-1} E_0^{\omega}(e^{s_3 \gamma_{\ell-1}^+}, X_{\gamma_{\ell-1}^+} = x) E_x^{\omega}(e^{s_3 \gamma_\ell^+}).$$

But if $x \cdot \hat{u} > \ell - 1$, then $E_x^{\omega}(e^{s_3 \gamma_\ell^+}) \leq E_0^{T_x \omega}(e^{s_3 \gamma_1^+})$. Therefore by induction

$$E_0^{\omega}(e^{s_3 \gamma_\ell^+}) \leq (e^{s_2} J_{s_3})^\ell \qquad \text{for any integer } \ell \geq 0. \tag{7.11}$$

Next for an integer $r \geq 1$,

$$E_0^{\omega}(e^{s_4 \gamma_{M_r}^+}) = \sum_{\ell=0}^{\infty} E_0^{\omega}(e^{s_4 \gamma_\ell^+}, M_r = \ell) \leq \sum_{\ell=0}^{\infty} E_0^{\omega}(e^{2s_4 \gamma_\ell^+})^{1/2} P_0^{\omega}(M_r = \ell)^{1/2}$$

$$\leq C \sum_{\ell=0}^{\infty} (e^{s_2} J_{2s_4})^{\ell/2} (\mathbb{1}\{\ell < 3Mr|\hat{u}|\} + e^{-s_5 \ell})^{1/2} \leq C(e^{s_2} J_{2s_4})^{Cr},$$

for some C and for positive but small enough s_2, s_4, and s_5. In the last inequality above we used the fact that $e^{s_2} J_{2s_4}$ converges to 1 as first $s_4 \searrow 0$ and then $s_2 \searrow 0$. In the second-to-last inequality we used (3.2) to get the bound

$$\sum_{i=1}^{r} P_0^{\omega}(X_i \cdot \hat{u} \geq \ell) \leq \sum_{i=1}^{r} e^{-s\ell} e^{M|\hat{u}|si} \leq Ce^{-s_5 \ell} \quad \text{if } \ell \geq 3M|\hat{u}|r.$$

Above we assumed that the walk X starts at 0. Same bounds work for any $x \in \mathbb{V}_d$ because a shift orthogonal to $\hat{u}$ does not alter levels, in particular $P_x^{\omega}(M_r = \ell) = P_0^{T_x \omega}(M_r = \ell)$.

By this same observation we show that for all $x, y \in \mathbb{V}_d$

$$E_{x,y}^{\omega}(e^{s_4 \gamma_{\tilde{M}_r}^+}) \leq C(e^{s_2} J_{2s_4})^{Cr}$$

by repeating the earlier series of inequalities.

Using (3.1) and these estimates gives for $x, y \in \mathbb{V}_d$

$$
\begin{aligned}
P^\omega_{x,y}(\rho \geq n, \beta \wedge \tilde{\beta} < \infty) &= \sum_{r=1}^{\infty} P^\omega_{x,y}(\gamma^+_{M_r \vee \widetilde{M}_r} \geq n, \beta \wedge \tilde{\beta} = r) \\
&\leq e^{-s_4 n/2} \sum_{1 \leq r \leq \varepsilon n} E^\omega_x(e^{s_4 \gamma^+_{M_r}})^{1/2} E^\omega_{x,y}(e^{s_4 \gamma^+_{\widetilde{M}_r}})^{1/2} \\
&\quad + \sum_{r > \varepsilon n} \left(P^\omega_x\{X_r \cdot \hat{u} < x \cdot \hat{u}\} + P^\omega_y\{\widetilde{X}_r \cdot \hat{u} < y \cdot \hat{u}\} \right) \\
&\leq C \varepsilon n e^{-s_4 n/2} (e^{s_2} J_{2s_4})^{C\varepsilon n} + C(1 - s_6 \delta/2)^{\varepsilon n}.
\end{aligned}
$$

Taking $\varepsilon > 0$ small enough shows the existence of a constant $\eta_2 > 0$ such that for all $x, y \in \mathbb{V}_d$, $n \geq 1$, and $\mathbb{P}$-a.e. ω,

$$
P^\omega_{x,y}(\rho \geq n, \beta \wedge \tilde{\beta} < \infty) \leq C(1 - \eta_2)^n.
$$

Same bound works for $\tilde{\rho}$ also. We combine this with (7.2) to get a geometric tail bound for $\nu_1 \mathbb{1}\{\beta \wedge \tilde{\beta} < \infty\}$. Recall definition (7.8) and take $\varepsilon > 0$ small.

$$
\begin{aligned}
&P^\omega_{x,y}[\nu_1 \geq k, \, \beta \wedge \tilde{\beta} < \infty] \\
&\leq P^\omega_{x,y}[\rho \geq k/2, \, \beta \wedge \tilde{\beta} < \infty] + P^\omega_{x,y}[\beta \wedge \tilde{\beta} < \infty, \, |X_\rho \cdot \hat{u} - \widetilde{X}_{\tilde{\rho}} \cdot \hat{u}| > \varepsilon k] \\
&\quad + P^\omega_{x,y}[\gamma_L \circ \theta^{\rho, \tilde{\rho}} \geq k/2, \, \beta \wedge \tilde{\beta} < \infty, \, |X_\rho \cdot \hat{u} - \widetilde{X}_{\tilde{\rho}} \cdot \hat{u}| \leq \varepsilon k].
\end{aligned}
$$

On the right-hand side above we have an exponential bound for each of the three probabilities: the first probability gets it from the estimate immediately above, the second from a combination of that and (3.2), and the third from (7.2):

$$
\begin{aligned}
&P^\omega_{x,y}[\gamma_L \circ \theta^{\rho, \tilde{\rho}} \geq k/2, \, \beta \wedge \tilde{\beta} < \infty, \, |X_\rho \cdot \hat{u} - \widetilde{X}_{\tilde{\rho}} \cdot \hat{u}| \leq \varepsilon k] \\
&= E^\omega_{x,y}\left[\mathbb{1}\{\beta \wedge \tilde{\beta} < \infty, \, |X_\rho \cdot \hat{u} - \widetilde{X}_{\tilde{\rho}} \cdot \hat{u}| \leq \varepsilon k\} \, P^\omega_{X_\rho, \widetilde{X}_{\tilde{\rho}}}\{\gamma_L \geq k/2\} \right] \\
&\leq C e^{a_1 \varepsilon k - a_2 k/2}.
\end{aligned}
$$

The constants in the last bound above are those from (7.2), and we choose $\varepsilon < a_2/(2a_1)$. We have thus established that

$$
E^\omega_{x,y}(e^{s_7 \nu_1} \mathbb{1}\{\beta \wedge \tilde{\beta} < \infty\}) \leq \bar{J}_{s_7}
$$

for a small enough $s_7 > 0$, with $\bar{J}_s = C(1 - (1 - \eta_3)e^s)^{-1}$ and $\eta_3 > 0$.

To move from ν_1 to ν_k use the Markov property and induction:

$$
\begin{aligned}
&E^\omega_{x,y}(e^{s_7 \nu_k} \mathbb{1}\{\nu_k \vee \tilde{\nu}_k < \infty\}) \\
&= \sum_{z, \tilde{z}} E^\omega_{x,y}(e^{s_7 \nu_{k-1}} \mathbb{1}\{\nu_{k-1} \vee \tilde{\nu}_{k-1} < \infty, X_{\nu_{k-1}} = z, \widetilde{X}_{\tilde{\nu}_{k-1}} = \tilde{z}\}) \\
&\qquad \times \, E^\omega_{z, \tilde{z}}(e^{s_7 \nu_1} \mathbb{1}\{\beta \wedge \tilde{\beta} < \infty\}) \\
&\leq \bar{J}_{s_7} E^\omega_{x,y}(e^{s_7 \nu_{k-1}} \mathbb{1}\{\nu_{k-1} \vee \tilde{\nu}_{k-1} < \infty\}) \leq \cdots \leq \bar{J}_{s_7}^k.
\end{aligned}
$$

Next, use the Markov property at the joint stopping times $(\nu_k, \tilde{\nu}_k)$, (7.2), (3.7), and induction to derive

$$
\begin{aligned}
P_{x,y}^{\omega}(K \geq k) &\leq P_{x,y}^{\omega}(\nu_k \vee \tilde{\nu}_k < \infty) \\
&\leq \sum_{z,\tilde{z}} P_{x,y}^{\omega}(\nu_{k-1} \vee \tilde{\nu}_{k-1} < \infty, X_{\nu_{k-1}} = z, \widetilde{X}_{\tilde{\nu}_{k-1}} = \tilde{z}) P_{z,\tilde{z}}^{\omega}(\beta \wedge \tilde{\beta} < \infty) \\
&\leq (1 - \eta^2) P_{x,y}^{\omega}(\nu_{k-1} \vee \tilde{\nu}_{k-1} < \infty) \leq (1 - \eta^2)^k.
\end{aligned}
$$

Finally use the Cauchy-Schwarz and Chebyshev inequalities to write

$$
\begin{aligned}
P_{x,y}^{\omega}(\nu_K \geq n) &= \sum_{k \geq 1} P_{x,y}^{\omega}(\nu_k \geq n, K = k) \\
&\leq \sum_{k > \varepsilon n} (1 - \eta^2)^k + e^{-s_7 n} \sum_{1 \leq k \leq \varepsilon n} E_{x,y}^{\omega}(e^{s_7 \nu_k} \mathbb{I}\{\nu_k \vee \tilde{\nu}_k < \infty\}) \\
&\leq C(1 - \eta^2)^{\varepsilon n} + C \varepsilon n e^{-s_7 n} \bar{J}_{s_7}^{\varepsilon n}.
\end{aligned}
$$

Looking at the definition (7.9) of μ_1 we see that an exponential tail bound follows by applying (7.2) to the ν-part and by taking $\varepsilon > 0$ small enough in the last calculation above. Repeat the same argument for $\tilde{\mu}_1$ to conclude the proof of (7.10). $\qquad \square$

After these preliminaries define the sequence of common regeneration times by $\mu_0 = \tilde{\mu}_0 = 0$ and

$$
(\mu_{i+1}, \tilde{\mu}_{i+1}) = (\mu_i, \tilde{\mu}_i) + (\mu_1, \tilde{\mu}_1) \circ \theta^{\mu_i, \tilde{\mu}_i}. \tag{7.12}
$$

The next tasks are to identify suitable Markovian structures and to develop a coupling.

PROPOSITION 7.4. *The process* $(\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i})_{i \geq 1}$ *is a Markov chain on* $\mathbb{V}_d$ *with transition probability*

$$
q(x, y) = P_{0,x}[\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = y \,|\, \beta = \tilde{\beta} = \infty]. \tag{7.13}
$$

Note that the time-homogeneous Markov chain does not start from $\widetilde{X}_0 - X_0$ because the transition to $\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1}$ does not include the condition $\beta = \tilde{\beta} = \infty$.

Proof. Express the iteration of the common regeneration times as

$$
(\mu_i, \tilde{\mu}_i) = (\mu_{i-1}, \tilde{\mu}_{i-1}) + \big((\nu, \tilde{\nu}) + (\nu_K, \tilde{\nu}_K) \circ \theta^{\nu, \tilde{\nu}}\big) \circ \theta^{\mu_{i-1}, \tilde{\mu}_{i-1}}, \quad i \geq 1.
$$

Let K_i be the value of K at the ith iteration:

$$
K_i = K \circ \theta^{\nu, \tilde{\nu}} \circ \theta^{\mu_{i-1}, \tilde{\mu}_{i-1}}.
$$

Let $n \geq 2$ and $z_1, \ldots, z_n \in \mathbb{V}_d$. Write

$$P_{0,z}[\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \leq i \leq n] \tag{7.14}$$

$$= \sum_{(k_i, m_i, \tilde{m}_i, v_i, \tilde{v}_i)_{1 \leq i \leq n-1} \in \Psi} P_{0,z}\big[K_i = k_i, \, \mu_i = m_i, \, \tilde{\mu}_i = \tilde{m}_i,$$

$$X_{m_i} = v_i \text{ and } \widetilde{X}_{\tilde{m}_i} = \tilde{v}_i \text{ for } 1 \leq i \leq n-1,$$

$$(\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1}) \circ \theta^{m_{n-1}, \tilde{m}_{n-1}} = z_n\big].$$

Above Ψ is the set of vectors $(k_i, m_i, \tilde{m}_i, v_i, \tilde{v}_i)_{1 \leq i \leq n-1}$ such that k_i is nonnegative and $m_i, \tilde{m}_i, v_i \cdot \hat{u}$, and $\tilde{v}_i \cdot \hat{u}$ are all positive and strictly increasing in i, and $\tilde{v}_i - v_i = z_i$.

Define the events

$$A_{k,b,\tilde{b}} = \{\nu + \nu_k \circ \theta^{\nu,\tilde{\nu}} = b, \, \tilde{\nu} + \tilde{\nu}_k \circ \theta^{\nu,\tilde{\nu}} = \tilde{b}\}$$

and

$$B_{b,\tilde{b}} = \{X_j \cdot \hat{u} \geq X_0 \cdot \hat{u} \text{ for } 1 \leq j \leq b, \, \widetilde{X}_j \cdot \hat{u} \geq \widetilde{X}_0 \cdot \hat{u} \text{ for } 1 \leq j \leq \tilde{b}\}.$$

Let $m_0 = \tilde{m}_0 = 0$, $b_i = m_i - m_{i-1}$ and $\tilde{b}_i = \tilde{m}_i - \tilde{m}_{i-1}$. Rewrite the sum from above as

$$\sum_{(k_i, m_i, \tilde{m}_i, v_i, \tilde{v}_i)_{1 \leq i \leq n-1} \in \Psi} E_{0,z}\Big[\prod_{i=1}^{n-1} \mathbb{I}\{A_{k_i, b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}}$$

$$\times \prod_{i=2}^{n-1} \mathbb{I}\{B_{b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}}, \, X_{m_i} = v_i \text{ and } \widetilde{X}_{\tilde{m}_i} = \tilde{v}_i \text{ for } 1 \leq i \leq n-1,$$

$$\beta \circ \theta^{m_{n-1}} = \tilde{\beta} \circ \theta^{\tilde{m}_{n-1}} = \infty, \, (\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1}) \circ \theta^{m_{n-1}, \tilde{m}_{n-1}} = z_n\Big].$$

Next restart the walks at times $(m_{n-1}, \tilde{m}_{n-1})$ to turn the sum into the following.

$$\sum_{(k_i, m_i, \tilde{m}_i, v_i, \tilde{v}_i)_{1 \leq i \leq n-1} \in \Psi} \mathbb{E}\bigg\{ E_{0,z}^{\omega}\Big[\prod_{i=1}^{n-1} \mathbb{I}\{A_{k_i, b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}}$$

$$\times \prod_{i=2}^{n-1} \mathbb{I}\{B_{b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}}, \, X_{m_i} = v_i \text{ and } \widetilde{X}_{\tilde{m}_i} = \tilde{v}_i \text{ for } 1 \leq i \leq n-1\Big]$$

$$\times P_{v_{n-1}, \tilde{v}_{n-1}}^{\omega}\Big[\beta = \tilde{\beta} = \infty, \, \widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n\Big] \bigg\}.$$

Inside the outermost braces the events in the first quenched expectation force the level

$$\ell = X_{m_{n-1}} \cdot \hat{u} = v_{n-1} \cdot \hat{u} = \widetilde{X}_{\tilde{m}_{n-1}} \cdot \hat{u} = \tilde{v}_{n-1} \cdot \hat{u}$$

to be a new maximal level for both walks. Consequently the first quenched expectation is a function of $\{\omega_x : x \cdot \hat{u} < \ell\}$ while the last quenched probability is a function of $\{\omega_x : x \cdot \hat{u} \geq \ell\}$. By independence of the environments, the sum becomes

$$\sum_{(k_i, m_i, \tilde{m}_i, v_i, \tilde{v}_i)_{1 \leq i \leq n-1} \in \Psi} E_{0,z} \Big[\prod_{i=1}^{n-1} \mathbb{1}\{A_{k_i, b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}} \tag{7.15}$$

$$\times \prod_{i=2}^{n-1} \mathbb{1}\{B_{b_i, \tilde{b}_i}\} \circ \theta^{m_{i-1}, \tilde{m}_{i-1}}, \, X_{m_i} = v_i \text{ and } \widetilde{X}_{\tilde{m}_i} = \tilde{v}_i \text{ for } 1 \leq i \leq n-1 \Big]$$

$$\times P_{v_{n-1}, \tilde{v}_{n-1}} \Big[\beta = \tilde{\beta} = \infty, \, \widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n \Big].$$

By a shift and a conditioning the last probability transforms as follows.

$$P_{v_{n-1}, \tilde{v}_{n-1}} \Big[\beta = \tilde{\beta} = \infty, \, \widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n \Big]$$
$$= P_{0, z_{n-1}} \Big[\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n \, \big| \, \beta = \tilde{\beta} = \infty \Big] P_{v_{n-1}, \tilde{v}_{n-1}} \Big[\beta = \tilde{\beta} = \infty \Big]$$
$$= q(z_{n-1}, z_n) P_{v_{n-1}, \tilde{v}_{n-1}} \Big[\beta = \tilde{\beta} = \infty \Big].$$

Now reverse the above use of independence to put the probability

$$P_{v_{n-1}, \tilde{v}_{n-1}} [\beta = \tilde{\beta} = \infty]$$

back together with the expectation (7.15). Inside this expectation this furnishes the event $\beta \circ \theta^{m_{n-1}} = \tilde{\beta} \circ \theta^{\tilde{m}_{n-1}} = \infty$ and with this the union of the entire collection of events turns back into $\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i$ for $1 \leq i \leq n-1$. Going back to the beginning on line (7.14) we see that we have now shown

$$P_{0,z}[\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \leq i \leq n]$$
$$= P_{0,z}[\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \leq i \leq n-1] q(z_{n-1}, z_n).$$

Continue by induction. $\square$

The Markov chain $Y_k = \widetilde{X}_{\tilde{\mu}_k} - X_{\mu_k}$ will be compared to a random walk obtained by performing the same construction of joint regeneration times to two independent walks in independent environments. To indicate the difference in construction we change notation. Let the pair of walks $(X, \bar{X})$ obey $P_0 \otimes P_z$ with $z \in \mathbb{V}_d$, and denote the first backtracking time of the $\bar{X}$ walk by $\bar{\beta} = \inf\{n \geq 1 : \bar{X}_n \cdot \hat{u} < \bar{X}_0 \cdot \hat{u}\}$. Construct the common regeneration times $(\rho_k, \bar{\rho}_k)_{k \geq 1}$ for $(X, \bar{X})$ by the same recipe [(7.8), (7.9) and (7.12)] as was used to construct $(\mu_k, \tilde{\mu}_k)_{k \geq 1}$ for $(X, \widetilde{X})$. Define $\bar{Y}_k = \bar{X}_{\bar{\rho}_k} - X_{\rho_k}$. An analogue of the previous proposition, which we will not spell out, shows that $(\bar{Y}_k)_{k \geq 1}$ is a Markov chain with transition

$$\bar{q}(x, y) = P_0 \otimes P_x[\bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y \, | \, \beta = \bar{\beta} = \infty]. \tag{7.16}$$

In the next two proofs we make use of the following decomposition. Suppose $x \cdot \hat{u} = y \cdot \hat{u} = 0$, and let (x_1, y_1) be another pair of points on a common, higher level: $x_1 \cdot \hat{u} = y_1 \cdot \hat{u} = \ell > 0$. Then we can write

$$\{(X_0, \widetilde{X}_0) = (x, y),\, \beta = \tilde{\beta} = \infty,\, (X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (x_1, y_1)\}$$
$$= \bigcup_{(\gamma, \tilde{\gamma})} \{X_{0,n(\gamma)} = \gamma,\, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma},\, \beta \circ \theta^{n(\gamma)} = \tilde{\beta} \circ \theta^{n(\tilde{\gamma})} = \infty\}. \qquad (7.17)$$

Here $(\gamma, \tilde{\gamma})$ range over all pairs of paths that connect (x, y) to (x_1, y_1), that stay between levels 0 and $\ell - 1$ before the final points, and for which a common regeneration fails at all levels before ℓ. $n(\gamma)$ is the index of the final point along the path, so for example $\gamma = (x = z_0, z_1, \ldots, z_{n(\gamma)-1}, z_{n(\gamma)} = x_1)$.

PROPOSITION 7.5. *The process $(\bar{Y}_k)_{k \geq 1}$ is a symmetric random walk on $\mathbb{V}_d$ and its transition probability satisfies*

$$\bar{q}(x, y) = \bar{q}(0, y - x) = \bar{q}(0, x - y) = P_0 \otimes P_0[\bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y - x \,|\, \beta = \bar{\beta} = \infty].$$

Proof. It remains to show that for independent $(X, \bar{X})$ the transition (7.16) reduces to a symmetric random walk. This becomes obvious once probabilities are decomposed into sums over paths because the events of interest are insensitive to shifts by $z \in \mathbb{V}_d$.

$$P_0 \otimes P_x[\beta = \bar{\beta} = \infty,\, \bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y]$$
$$= \sum_w P_0 \otimes P_x[\beta = \bar{\beta} = \infty,\, X_{\rho_1} = w,\, \bar{X}_{\bar{\rho}_1} = y + w]$$
$$= \sum_w \sum_{(\gamma, \bar{\gamma})} P_0[X_{0,n(\gamma)} = \gamma,\, \beta \circ \theta^{n(\gamma)} = \infty] P_x[X_{0,n(\bar{\gamma})} = \bar{\gamma},\, \beta \circ \theta^{n(\bar{\gamma})} = \infty] \qquad (7.18)$$
$$= \sum_w \sum_{(\gamma, \bar{\gamma})} P_0[X_{0,n(\gamma)} = \gamma] P_x[X_{0,n(\bar{\gamma})} = \bar{\gamma}] \big(P_0[\beta = \infty]\big)^2.$$

Above we used the decomposition idea from (7.17). Here $(\gamma, \bar{\gamma})$ range over the appropriate class of pairs of paths in $\mathbb{Z}^d$ such that γ goes from 0 to w and $\bar{\gamma}$ goes from x to $y + w$. The independence for the last equality above comes from noticing that the quenched probabilities $P_0^\omega[X_{0,n(\gamma)} = \gamma]$ and $P_w^\omega[\beta = \infty]$ depend on independent collections of environments.

The probabilities on the last line of (7.18) are not changed if each pair $(\gamma, \bar{\gamma})$ is replaced by $(\gamma, \gamma') = (\gamma, \bar{\gamma} - x)$. These pairs connect $(0, 0)$ to $(w, y - x + w)$. Because $x \in \mathbb{V}_d$ satisfies $x \cdot \hat{u} = 0$, the shift has not changed regeneration levels. This shift turns $P_x[X_{0,n(\bar{\gamma})} = \bar{\gamma}]$ on the last line of (7.18) into $P_0[X_{0,n(\gamma')} = \gamma']$. We can reverse the steps in (7.18) to arrive at the probability

$$P_0 \otimes P_0[\beta = \bar{\beta} = \infty,\, \bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y - x].$$

This proves $\bar{q}(x, y) = \bar{q}(0, y - x)$.

Once both walks start at 0 it is immaterial which is labeled X and which $\bar{X}$, hence symmetry holds. $\qquad\square$

It will be useful to know that $\bar{q}$ inherits all possible transitions from q.

LEMMA 7.6. *If* $q(z, w) > 0$ *then also* $\bar{q}(z, w) > 0$.

Proof. By the decomposition from (7.17) we can express

$$P_{x,y}[(X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (x_1, y_1)|\beta = \tilde{\beta} = \infty] = \sum_{(\gamma, \tilde{\gamma})} \frac{\mathbb{E}P^\omega[\gamma]P^\omega[\tilde{\gamma}]P^\omega_{x_1}[\beta = \infty]P^\omega_{y_1}[\beta = \infty]}{P_{x,y}[\beta = \tilde{\beta} = \infty]}.$$

If this probability is positive, then at least one pair $(\gamma, \tilde{\gamma})$ satisfies $\mathbb{E}P^\omega[\gamma]P^\omega[\tilde{\gamma}] > 0$. This implies that $P[\gamma]P[\tilde{\gamma}] > 0$ so that also

$$P_x \otimes P_y[(X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (x_1, y_1)|\beta = \tilde{\beta} = \infty] > 0. \qquad\square$$

In the sequel we detach the notations $Y = (Y_k)$ and $\bar{Y} = (\bar{Y}_k)$ from their original definitions in terms of the walks X, $\widetilde{X}$ and $\bar{X}$, and use (Y_k) and $(\bar{Y}_k)$ to denote canonical Markov chains with transitions q and $\bar{q}$. Now we construct a coupling.

PROPOSITION 7.7. *The single-step transitions* $q(x, y)$ *for* Y *and* $\bar{q}(x, y)$ *for* $\bar{Y}$ *can be coupled in such a way that, when the processes start from a common state* x,

$$P_{x,x}[Y_1 \neq \bar{Y}_1] \leq Ce^{-\alpha_1|x|}$$

for all $x \in \mathbb{V}_d$. *Here* C *and* α_1 *are finite positive constants independent of* x.

Proof. We start by constructing a coupling of three walks $(X, \widetilde{X}, \bar{X})$ such that the pair $(X, \widetilde{X})$ has distribution $P_{x,y}$ and the pair $(X, \bar{X})$ has distribution $P_x \otimes P_y$.

First let $(X, \widetilde{X})$ be two independent walks in a common environment ω as before. Let $\bar{\omega}$ be an environment independent of ω. Define the walk $\bar{X}$ as follows. Initially $\bar{X}_0 = \widetilde{X}_0$. On the sites $\{X_k : 0 \leq k < \infty\}$ $\bar{X}$ obeys environment $\bar{\omega}$, and on all other sites $\bar{X}$ obeys ω. $\bar{X}$ is coupled to agree with $\widetilde{X}$ until the time

$$T = \inf\{n \geq 0 : \bar{X}_n \in \{X_k : 0 \leq k < \infty\}\}$$

it hits the path of X.

The coupling between $\bar{X}$ and $\widetilde{X}$ can be achieved simply as follows. Given ω and $\bar{\omega}$, for each x create two independent i.i.d. sequences $(z^x_k)_{k \geq 1}$ and $(\bar{z}^x_k)_{k \geq 1}$ with distributions

$$Q^{\omega, \bar{\omega}}[z^x_k = y] = \pi_{x, x+y}(\omega) \quad \text{and} \quad Q^{\omega, \bar{\omega}}[\bar{z}^x_k = y] = \pi_{x, x+y}(\bar{\omega}).$$

Do this independently at each x. Each time the $\widetilde{X}$-walk visits state x, it uses a new z^x_k variable as its next step, and never reuses the same z^x_k again. The $\bar{X}$ walk operates the same way except that it uses the variables $\bar{z}^x_k$ when $x \in \{X_k\}$ and the z^x_k variables when $x \notin \{X_k\}$. Now $\bar{X}$ and $\widetilde{X}$ follow the same steps z^x_k until $\bar{X}$ hits the set $\{X_k\}$.

It is intuitively obvious that the walks X and $\bar{X}$ are independent because they never use the same environment. The following calculation verifies this. Let $X_0 = x_0 = x$

and $\widetilde{X} = \bar{X} = y_0 = y$ be the initial states, and $\mathbf{P}_{x,y}$ the joint measure created by the coupling. Fix finite vectors $x_{0,n} = (x_0, \ldots, x_n)$ and $y_{0,n} = (y_0, \ldots, y_n)$ and recall also the notation $X_{0,n} = (X_0, \ldots, X_n)$.

The description of the coupling tells us to start as follows.

$$\mathbf{P}_{x,y}[X_{0,n} = x_{0,n}, \bar{X}_{0,n} = y_{0,n}] = \int \mathbb{P}(d\omega) \int \mathbb{P}(d\bar{\omega}) \int P_x^\omega(dz_{0,\infty}) \mathbb{1}\{z_{0,n} = x_{0,n}\}$$

$$\times \prod_{i:y_i \notin \{z_k : 0 \le k < \infty\}} \pi_{y_i, y_{i+1}}(\omega) \cdot \prod_{i:y_i \in \{z_k : 0 \le k < \infty\}} \pi_{y_i, y_{i+1}}(\bar{\omega})$$

[by dominated convergence]

$$= \lim_{N \to \infty} \int \mathbb{P}(d\omega) \int \mathbb{P}(d\bar{\omega}) \int P_x^\omega(dz_{0,N}) \mathbb{1}\{z_{0,n} = x_{0,n}\}$$

$$\times \prod_{i:y_i \notin \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega) \cdot \prod_{i:y_i \in \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega})$$

$$= \lim_{N \to \infty} \sum_{z_{0,N} : z_{0,n} = x_{0,n}} \int \mathbb{P}(d\omega) P_x^\omega[X_{0,N} = z_{0,N}] \prod_{i:y_i \notin \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega)$$

$$\times \int \mathbb{P}(d\bar{\omega}) \prod_{i:y_i \in \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega})$$

[by independence of the two functions of ω]

$$= \lim_{N \to \infty} \sum_{z_{0,N} : z_{0,n} = x_{0,n}} \int \mathbb{P}(d\omega) P_x^\omega[X_{0,N} = z_{0,N}] \int \mathbb{P}(d\omega) \prod_{i:y_i \notin \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega)$$

$$\times \int \mathbb{P}(d\bar{\omega}) \prod_{i:y_i \in \{z_k : 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega})$$

$$= P_x[X_{0,n} = x_{0,n}] \cdot P_y[X_{0,n} = y_{0,n}].$$

Thus at this point the coupled pairs $(X, \widetilde{X})$ and $(X, \bar{X})$ have the desired marginals $P_{x,y}$ and $P_x \otimes P_y$.

Next construct the common regeneration times $(\mu_1, \tilde{\mu}_1)$ for $(X, \widetilde{X})$ and $(\rho_1, \bar{\rho}_1)$ for $(X, \bar{X})$ by the earlier recipes. Define two pairs of walks stopped at their common regeneration times:

$$(\Gamma, \bar{\Gamma}) \equiv \left((X_{0,\mu_1}, \widetilde{X}_{0,\tilde{\mu}_1}), (X_{0,\rho_1}, \bar{X}_{0,\bar{\rho}_1}) \right). \tag{7.19}$$

Suppose the sets $X_{[0,\mu_1 \vee \rho_1)}$ and $\widetilde{X}_{[0,\tilde{\mu}_1 \vee \bar{\rho}_1)}$ do not intersect. Then the construction implies that the path $\bar{X}_{0,\tilde{\mu}_1 \vee \bar{\rho}_1}$ agrees with $\widetilde{X}_{0,\tilde{\mu}_1 \vee \bar{\rho}_1}$, and this forces the equalities $(\mu_1, \tilde{\mu}_1) = (\rho_1, \bar{\rho}_1)$ and $(X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (X_{\rho_1}, \bar{X}_{\bar{\rho}_1})$. We insert an estimate on this event.

LEMMA 7.8. *There exist constants $0 < C, s < \infty$ such that, for all $x, y \in \mathbb{V}_d$ and $\mathbb{P}$-a.e. ω,*

$$P^\omega_{x,y}(X_{[0,\,\mu_1 \vee \rho_1)} \cap \widetilde{X}_{[0,\,\tilde{\mu}_1 \vee \bar{\rho}_1)} \neq \emptyset) \leq Ce^{-s|x-y|}. \tag{7.20}$$

Proof. Write

$$\begin{aligned}
P^\omega_{x,y}(X_{[0,\,\mu_1 \vee \rho_1)} \cap \widetilde{X}_{[0,\,\tilde{\mu}_1 \vee \bar{\rho}_1)} \neq \emptyset) \leq{} & P^\omega_{x,y}(\mu_1 \vee \tilde{\mu}_1 \vee \rho_1 \vee \bar{\rho}_1 > \varepsilon|x-y|) \\
& + P^\omega_x\big(\max_{1 \leq i \leq \varepsilon|x-y|} |X_i - x| \geq |x-y|/2 \big) \\
& + P^\omega_y\big(\max_{1 \leq i \leq \varepsilon|x-y|} |X_i - y| \geq |x-y|/2 \big).
\end{aligned}$$

By (7.10) and its analogue for $(\rho_1, \bar{\rho}_1)$ the first term on the right-hand-side decays exponentially in $|x - y|$. Using (3.2) the second and third terms are bounded by $\varepsilon|x - y|e^{-s|x-y|/2}e^{\varepsilon s|x-y|M}$, for $s > 0$ small enough. Choosing $\varepsilon > 0$ small enough finishes the proof. $\qquad \square$

From (7.20) we obtain

$$\mathbf{P}_{x,y}\big[(X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) \neq (X_{\rho_1}, \bar{X}_{\bar{\rho}_1}) \big] \leq \mathbf{P}_{x,y}\big[\Gamma \neq \bar{\Gamma} \big] \leq Ce^{-s|x-y|}. \tag{7.21}$$

But we are not finished yet: it remains to include the conditioning on no back-tracking. For this purpose generate an i.i.d. sequence $(X^{(m)}, \widetilde{X}^{(m)}, \bar{X}^{(m)})_{m \geq 1}$, each triple constructed as above. Continue to write $\mathbf{P}_{x,y}$ for the probability measure of the entire sequence. Let M be the first m such that the paths $(X^{(m)}, \widetilde{X}^{(m)})$ do not backtrack, which means that

$$X^{(m)}_k \cdot \hat{u} \geq X^{(m)}_0 \cdot \hat{u} \text{ and } \widetilde{X}^{(m)}_k \cdot \hat{u} \geq \widetilde{X}^{(m)}_0 \cdot \hat{u} \text{ for all } k \geq 1.$$

Similarly define $\bar{M}$ for $(X^{(m)}, \bar{X}^{(m)})_{m \geq 1}$. M and $\bar{M}$ are stochastically bounded by geometric random variables by (3.7).

The pair of walks $(X^{(M)}, \widetilde{X}^{(M)})$ is now distributed as a pair of walks under the measure $P_{x,y}[\,\cdot\,|\beta = \tilde{\beta} = \infty]$, while $(X^{(\bar{M})}, \bar{X}^{(\bar{M})})$ is distributed as a pair of walks under $P_x \otimes P_y[\,\cdot\,|\beta = \bar{\beta} = \infty]$.

Let also again

$$\Gamma^{(m)} = (X^{(m)}_{0,\,\mu^{(m)}_1}, \widetilde{X}^{(m)}_{0,\,\tilde{\mu}^{(m)}_1}) \quad \text{and} \quad \bar{\Gamma}^{(m)} = (X^{(m)}_{0,\,\rho^{(m)}_1}, \bar{X}^{(m)}_{0,\,\bar{\rho}^{(m)}_1})$$

be the pairs of paths run up to their common regeneration times. Consider the two pairs of paths $(\Gamma^{(M)}, \bar{\Gamma}^{(\bar{M})})$ chosen by the random indices $(M, \bar{M})$. We insert one more lemma.

LEMMA 7.9. *For $s > 0$ as above, and a new constant $0 < C < \infty$,*

$$\mathbf{P}_{x,y}\big[\Gamma^{(M)} \neq \bar{\Gamma}^{(\bar{M})} \big] \leq Ce^{-s|x-y|/2}. \tag{7.22}$$

Proof. Let $\mathcal{A}_m$ be the event that the walks $\widetilde{X}^{(m)}$ and $\bar{X}^{(m)}$ agree up to the maximum $\tilde{\mu}_1^{(m)} \vee \bar{\rho}_1^{(m)}$ of their regeneration times. The equalities $M = \bar{M}$ and $\Gamma^{(M)} = \bar{\Gamma}^{(\bar{M})}$ are a consequence of the event $\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_M$, for the following reason. As pointed out earlier, on the event $\mathcal{A}_m$ we have the equality of the regeneration times $\tilde{\mu}_1^{(m)} = \bar{\rho}_1^{(m)}$ and of the stopped paths $\widetilde{X}_{0,\tilde{\mu}_1^{(m)}}^{(m)} = \bar{X}_{0,\bar{\rho}_1^{(m)}}^{(m)}$. By definition, these walks do not backtrack after the regeneration time. Since the walks $\widetilde{X}^{(m)}$ and $\bar{X}^{(m)}$ agree up to this time, they must backtrack or fail to backtrack together. If this is true for each $m = 1, \ldots, M$, it forces $\bar{M} = M$, since the other factor in deciding M and $\bar{M}$ are the paths $X^{(m)}$ that are common to both. And since the paths agree up to the regeneration times, we have $\Gamma^{(M)} = \bar{\Gamma}^{(\bar{M})}$.

Estimate (7.22) follows:

$$\mathbf{P}_{x,y}\left[\Gamma^{(M)} \neq \bar{\Gamma}^{(\bar{M})}\right] \leq \mathbf{P}_{x,y}\left[\mathcal{A}_1^c \cup \cdots \cup \mathcal{A}_M^c\right]$$

$$\leq \sum_{m=1}^\infty \mathbf{P}_{x,y}[M \geq m,\, \mathcal{A}_m^c] \leq \sum_{m=1}^\infty \left(\mathbf{P}_{x,y}[M \geq m]\right)^{1/2} \left(\mathbf{P}_{x,y}[\mathcal{A}_m^c]\right)^{1/2}$$

$$\leq Ce^{-s|x-y|/2}.$$

The last step comes from the estimate in (7.20) for each $\mathcal{A}_m^c$ and the geometric bound on M.   □

We are ready to finish the proof of Proposition 7.7. To create initial conditions $Y_0 = \bar{Y}_0 = x$ take initial states $(X_0^{(m)}, \widetilde{X}_0^{(m)}) = (X_0^{(m)}, \bar{X}_0^{(m)}) = (0, x)$. Let the final outcome of the coupling be the pair

$$(Y_1, \bar{Y}_1) = \left(\widetilde{X}_{\tilde{\mu}_1^{(M)}}^{(M)} - X_{\mu_1^{(M)}}^{(M)}, \, \bar{X}_{\bar{\rho}_1^{(\bar{M})}}^{(\bar{M})} - X_{\rho_1^{(\bar{M})}}^{(\bar{M})}\right)$$

under the measure $\mathbf{P}_{0,x}$. The marginal distributions of Y_1 and $\bar{Y}_1$ are correct [namely, given by the transitions (7.13) and (7.16)] because, as argued above, the pairs of walks themselves have the right marginal distributions. The event $\Gamma^{(M)} = \bar{\Gamma}^{(\bar{M})}$ implies $Y_1 = \bar{Y}_1$, so estimate (7.22) gives the bound claimed in Proposition 7.7.   □

The construction of the Markov chain is complete, and we return to the main development of the proof. It remains to prove a sublinear bound on the expected number $E_{0,0}|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|$ of common points of two independent walks in a common environment. Utilizing the common regeneration times, write

$$E_{0,0}|X_{[0,n)} \cap \widetilde{X}_{[0,n)}| \leq \sum_{i=0}^{n-1} E_{0,0}|X_{[\mu_i, \mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i, \tilde{\mu}_{i+1})}|. \tag{7.23}$$

The term $i = 0$ is a finite constant by bound (7.10) because the number of common points is bounded by the number μ_1 of steps. For each $0 < i < n$ apply a decomposition into pairs of paths from $(0,0)$ to given points (x_1, y_1) in the style of (7.17):

$(\gamma, \tilde{\gamma})$ are the pairs of paths with the property that

$$\bigcup_{(\gamma, \tilde{\gamma})} \{X_{0,n(\gamma)} = \gamma, \, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}, \, \beta \circ \theta^{n(\gamma)} = \tilde{\beta} \circ \theta^{n(\tilde{\gamma})} = \infty\}$$

$$= \{X_0 = \widetilde{X}_0 = 0, \, X_{\mu_i} = x_1, \, \widetilde{X}_{\tilde{\mu}_i} = y_1\}.$$

Each term $i > 0$ in (7.23) we rearrange as follows.

$$E_{0,0}|X_{[\mu_i,\mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i,\tilde{\mu}_{i+1})}|$$

$$= \sum_{x_1,y_1} \sum_{(\gamma,\tilde{\gamma})} \mathbb{E} P_{0,0}^\omega[X_{0,n(\gamma)} = \gamma, \, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}]$$

$$\times E_{x_1,y_1}^\omega(\mathbb{1}\{\beta = \tilde{\beta} = \infty\}|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}|)$$

$$= \sum_{x_1,y_1} \sum_{(\gamma,\tilde{\gamma})} \mathbb{E} P_{0,0}^\omega[X_{0,n(\gamma)} = \gamma, \, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}] P_{x_1,y_1}^\omega[\beta = \tilde{\beta} = \infty]$$

$$\times E_{x_1,y_1}^\omega(|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty)$$

$$= \sum_{x_1,y_1} \mathbb{E} P_{0,0}^\omega[X_{\mu_i} = x_1, \, \widetilde{X}_{\tilde{\mu}_i} = y_1] E_{x_1,y_1}^\omega(|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty).$$

The last conditional quenched expectation above is handled by estimates (3.7), (7.10), (7.20) and Schwarz inequality:

$$E_{x_1,y_1}^\omega(|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty) \le \eta^{-2} E_{x_1,y_1}^\omega(|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}|)$$

$$\le \eta^{-2} E_{x_1,y_1}^\omega(\mu_1 \cdot \mathbb{1}\{X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)} \ne \emptyset\})$$

$$\le \eta^{-2} \left(E_{x_1,y_1}^\omega[\mu_1^2]\right)^{1/2} \left(P_{x_1,y_1}^\omega\{X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)} \ne \emptyset\}\right)^{1/2}$$

$$\le C e^{-s|x_1 - y_1|/2}.$$

Define $h(x) = C e^{-s|x|/2}$, insert the last bound back up, and appeal to the Markov property established in Proposition 7.4:

$$E_{0,0}|X_{[\mu_i,\mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i,\tilde{\mu}_{i+1})}| \le E_{0,0}\left[h(\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i})\right]$$

$$= \sum_x P_{0,0}[\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = x] \sum_y q^{i-1}(x,y) h(y).$$

In order to apply Theorem A.1 from the Appendix, we check its hypotheses in the next lemma. Assumption (1.2) enters here for the first and only time.

LEMMA 7.10. *The Markov chain* $(Y_k)_{k \ge 0}$ *with transition* $q(x,y)$ *and the symmetric random walk* $(\bar{Y}_k)_{k \ge 0}$ *with transition* $\bar{q}(x,y)$ *satisfy assumptions* (A.i), (A.ii) *and* (A.iii) *stated in the beginning of the Appendix.*

Proof. From Lemma 7.3 and (3.2) we get moment bounds

$$E_{0,x}|\bar{X}_{\bar{\rho}_k}|^m + E_{0,x}|X_{\rho_k}|^m < \infty$$

for any power $m < \infty$. This gives assumption (A.i), namely that $E_0|\bar{Y}_1|^3 < \infty$. The second part of assumption (A.ii) comes from Lemma 7.6. Assumption (A.iii) comes from Proposition 7.7.

The only part that needs work is the first part of assumption (A.ii). We show that it follows from part (1.2) of Hypothesis (R). By (1.2) and non-nestling (N) there exist two non-zero vectors $y \neq z$ such that $z \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0,y}\pi_{0,z} > 0$. Now we have a number of cases to consider. In each case we should describe an event that gives $Y_1 - Y_0$ a particular nonzero value and whose probability is bounded away from zero, uniformly over $x = Y_0$.

Case 1: y is noncollinear with z. The sign of $y \cdot \hat{u}$ gives three subcases. We do the trickiest one explicitly. Assume $y \cdot \hat{u} < 0$. Find the smallest positive integer b such that $(y + bz) \cdot \hat{u} > 0$. Then find the minimal positive integers k, m such that $k(y + bz) \cdot \hat{u} = mz \cdot \hat{u}$. Below P_x is the path measure of the Markov chain (Y_k) and then $P_{0,x}$ the measure of the walks $(X, \widetilde{X})$ as before.

$$\begin{aligned}
P_x\{Y_1 - Y_0 &= ky + (kb - m)z\} \\
&\geq P_{0,x}\big\{\widetilde{X}_{\tilde{\mu}_1} = x + ky + (k+1)bz\,,\ X_{\mu_1} = (m+b)z\,,\ \beta = \tilde{\beta} = \infty\big\} \\
&\geq \mathbb{E}\Big[P_0^{T_x\omega}\{X_{i(b+1)+1} = i(y+bz) + z, \ldots, X_{i(b+1)+b} = i(y+bz) + bz, \\
&\qquad\qquad X_{(i+1)(b+1)} = (i+1)(y+bz) \quad \text{for } 0 \leq i \leq k-1, \ \text{and then} \\
&\qquad\qquad X_{k(b+1)+1} = k(y+bz) + z, \ldots, X_{k(b+1)+b} = k(y+bz) + bz \} \\
&\qquad\times P_0^\omega\{ X_1 = z\,,\ X_2 = 2z\,,\ldots,\ X_{m+b} = (m+b)z \} \\
&\qquad\times P_{x+ky+(k+1)bz}^\omega\{\beta = \infty\} P_{(m+b)z}^\omega\{\beta = \infty\}\Big].
\end{aligned}$$

Regardless of possible intersections of the paths, assumption (1.2) and inequality (3.7) imply that the quantity above has a positive lower bound that is independent of x. The assumption that y, z are nonzero and noncollinear ensures that $ky + (kb-m)z \neq 0$.

Case 2: y is collinear with z. Then there is a vector $w \notin \mathbb{R}z$ such that $\mathbb{E}\pi_{0,w} > 0$. If $w \cdot \hat{u} \leq 0$, then by Hypothesis (N) there exists u such that $u \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0,w}\pi_{0,u} > 0$. If u is collinear with z, then replacing z by u and y by w puts us back in Case 1. So, replacing w by u if necessary, we can assume that $w \cdot \hat{u} > 0$. We have four subcases, depending on whether $x = 0$ or not and $y \cdot \hat{u} < 0$ or not.

(2.a) The case $x \neq 0$ is resolved simply by taking paths consisting of only w-steps for one walk and only z-steps for the other, until they meet on a common level and then never backtrack.

(2.b) The case $y \cdot \hat{u} > 0$ corresponds to Case 3 in the proof of [11, Lemma 5.5].

(2.c) The only case left is $x = 0$ and $y \cdot \hat{u} < 0$. Let b and c be the smallest positive integers such that $(y + bw) \cdot \hat{u} \geq 0$ and $(y + cz) \cdot \hat{u} > 0$. Choose minimal positive

integeres $m \geq b$ and $n > c$ such that $m(w \cdot \hat{u}) = n(z \cdot \hat{u})$. Then,

$$P_0\{Y_1 - Y_0 = nz - mw\}$$
$$\geq P_{0,0}\{\widetilde{X}_{\tilde{\mu}_1} = y + bw + nz, X_{\mu_1} = y + (b+m)w\}$$
$$\geq \mathbb{E}\Big[P_0^\omega\{X_i = iw \text{ for } 1 \leq i \leq b \text{ and } X_{b+1+j} = y + (b+j)w \text{ for } 0 \leq j \leq m\}$$
$$\times P_0^\omega\{X_i = iw \text{ for } 0 \leq i \leq b, X_{b+1} = bw + z \text{ and then}$$
$$X_{b+1+j} = y + bw + jz \text{ for } 1 \leq j \leq n\}$$
$$\times P_{y+(b+m)w}^\omega(\beta = \infty)P_{y+bw+nz}^\omega(\beta = \infty)\Big].$$

Since w and z are noncollinear, $mw \neq nz$. For the same reason, w-steps are always taken at points not visited before. This makes the above lower bound positive. By the choice of b and $z \cdot \hat{u} > 0$, neither walk dips below level 0.

We can see that the first common regeneration level for the two paths is $(y + bw + nz) \cdot \hat{u}$. The first walk backtracks from level $bw \cdot \hat{u}$ so this is not a common regeneration level. The second walk splits from the first walk at bw, takes a z-step up, and then backtracks using a y-step. So the common regeneration level can only be at or above level $(y + bw + (c+1)z) \cdot \hat{u}$. The fact that $n > c$ ensures that $(y + bw + nz) \cdot \hat{u}$ is high enough. The minimality of n ensures that this is the first such level. $\qquad\square$

Now that the assumptions have been checked, Theorem A.1 gives constants $0 < C < \infty$ and $0 < \eta < 1$ such that

$$\sum_{i=1}^{n-1}\sum_y q^{i-1}(x, y)h(y) \leq Cn^{1-\eta} \quad \text{for all } x \in \mathbb{V}_d \text{ and } n \geq 1.$$

Going back to (7.23) and collecting the bounds along the way gives the final estimate

$$E_{0,0}|X_{[0,n)} \cap \widetilde{X}_{[0,n)}| \leq Cn^{1-\eta}$$

for all $n \geq 1$. This is (6.2) which was earlier shown to imply condition (2.1) required by Theorem 2.1. Previous work in Sections 2 and 5 convert the CLT from Theorem 2.1 into the main result Theorem 1.1. The entire proof is complete, except for the Green function estimate furnished by the Appendix.

APPENDIX A. A GREEN FUNCTION TYPE BOUND

Let us write a d-vector in terms of coordinates as $x = (x^1, \ldots, x^d)$, and similarly for random vectors $X = (X^1, \ldots, X^d)$.

Let $Y = (Y_k)_{k \geq 0}$ be a Markov chain on $\mathbb{Z}^d$ with transition probability $q(x, y)$, and let $\bar{Y} = (\bar{Y}_k)_{k \geq 0}$ be a symmetric random walk on $\mathbb{Z}^d$ with transition probability $\bar{q}(x, y) = \bar{q}(y, x) = \bar{q}(0, y - x)$. Make the following assumptions.

(A.i) A third moment bound $E_0|\bar{Y}_1|^3 < \infty$.

(A.ii) Some uniform nondegeneracy: there is at least one index $j \in \{1, \ldots, d\}$ and a constant κ_0 such that the coordinate Y^j satisfies

$$P_x\{Y_1^j - Y_0^j \geq 1\} \geq \kappa_0 > 0 \quad \text{for all } x. \tag{A.1}$$

(The inequality ≥ 1 can be replaced by ≤ -1, the point is to assure that a cube is exited fast enough.) Furthermore, for every $i \in \{1, \ldots, d\}$, if the one-dimensional random walk $\bar{Y}^i$ is degenerate in the sense that $\bar{q}(0, y) = 0$ for $y^i \neq 0$, then so is the process Y^i in the sense that $q(x, y) = 0$ whenever $x^i \neq y^i$. In other words, any coordinate that can move in the Y chain somewhere in space can also move in the $\bar{Y}$ walk.

(A.iii) Most importantly, assume that for any initial state x the transitions q and $\bar{q}$ can be coupled so that

$$P_{x,x}[Y_1 \neq \bar{Y}_1] \leq Ce^{-\alpha_1 |x|}$$

where $0 < C, \alpha_1 < \infty$ are constants independent of x.

Throughout the section C will change value but α_1 remains the constant in the assumption above. Let h be a function on $\mathbb{Z}^d$ such that $0 \leq h(x) \leq Ce^{-\alpha_2 |x|}$ for constants $0 < \alpha_2, C < \infty$. This section is devoted to proving the following Green function type bound on the Markov chain.

THEOREM A.1. *There are constants $0 < C, \eta < \infty$ such that*

$$\sum_{k=0}^{n-1} E_z h(Y_k) = \sum_y h(y) \sum_{k=0}^{n-1} P_0(Y_k = y) \leq Cn^{1-\eta} \quad \text{for all } n \geq 1 \text{ and } z \in \mathbb{Z}^d.$$

To prove the estimate, we begin by discarding terms outside a cube of side $r = c_1 \log n$. Bounding probabilities crudely by 1 gives

$$\sum_{|y| > c_1 \log n} h(y) \sum_{k=0}^{n-1} P_z(Y_k = y) \leq n \sum_{|y| > c_1 \log n} h(y) \leq Cn \sum_{k > c_1 \log n} k^{d-1} e^{-\alpha_2 k}$$

$$\leq Cn \sum_{k > c_1 \log n} e^{-(\alpha_2/2)k} \leq Cne^{-(\alpha_2/2)c_1 \log n} \leq Cn^{1-\eta}$$

as long as n is large enough so that $k^{d-1} \leq e^{\alpha_2 k/2}$, and this works for any c_1.

Let

$$B = [-c_1 \log n, c_1 \log n]^d.$$

Since h is bounded, it now remains to show that

$$\sum_{k=0}^{n-1} P_z(Y_k \in B) \leq Cn^{1-\eta}. \tag{A.2}$$

For this we can assume $z \in B$ since accounting for the time to enter B for the first time can only improve the estimate.

Bound (A.2) will be achieved in two stages. First we show that the Markov chain Y does not stay in B longer than a time whose mean is a power of the size of B. Second, we show that often enough Y follows the random walk $\bar{Y}$ during its excursions outside B. The random walk excursions are long and thereby we obtain (A.2). Thus our first task is to construct a suitable coupling of Y and $\bar{Y}$.

LEMMA A.1. *Let* $\zeta = \inf\{n \geq 1 : \bar{Y} \in A\}$ *be the first entrance time of* $\bar{Y}$ *into some set* $A \subseteq \mathbb{Z}^d$. *Then we can couple* Y *and* $\bar{Y}$ *so that*

$$P_{x,x}[\ Y_k \neq \bar{Y}_k \ \text{for some}\ 1 \leq k \leq \zeta\ ] \leq CE_x \sum_{k=0}^{\zeta-1} e^{-\alpha_1|\bar{Y}_k|}.$$

The proof shows that the statement works also if $\zeta = \infty$ is possible, but we will not need this case.

Proof. For each state x create an i.i.d. sequence $(Z_k^x, \bar{Z}_k^x)_{k \geq 1}$ such that Z_k^x has distribution $q(x, x + \cdot)$, $\bar{Z}_k^x$ has distribution $\bar{q}(x, x + \cdot) = \bar{q}(0, \cdot)$, and each pair $(Z_k^x, \bar{Z}_k^x)$ is coupled so that $P(Z_k^x \neq \bar{Z}_k^x) \leq Ce^{-\alpha_1|x|}$. For distinct x these sequences are independent.

Construct the process $(Y_n, \bar{Y}_n)$ as follows: with counting measures

$$L_n(x) = \sum_{k=0}^{n} \mathbb{I}\{Y_k = x\} \quad \text{and} \quad \bar{L}_n(x) = \sum_{k=0}^{n} \mathbb{I}\{\bar{Y}_k = x\} \quad (n \geq 0)$$

and with initial point $(Y_0, \bar{Y}_0)$ given, define for $n \geq 1$

$$Y_n = Y_{n-1} + Z_{L_{n-1}(Y_{n-1})}^{Y_{n-1}} \quad \text{and} \quad \bar{Y}_n = \bar{Y}_{n-1} + \bar{Z}_{\bar{L}_{n-1}(\bar{Y}_{n-1})}^{\bar{Y}_{n-1}}.$$

In words, every time the chain Y visits a state x, it reads its next jump from a new variable Z_k^x which is then discarded and never used again. And similarly for $\bar{Y}$. This construction has the property that, if $Y_k = \bar{Y}_k$ for $0 \leq k \leq n$ with $Y_n = \bar{Y}_n = x$, then the next joint step is $(Z_k^x, \bar{Z}_k^x)$ for $k = L_n(x) = \bar{L}_n(x)$. In other words, given that the processes agree up to the present and reside together at x, the probability that they separate in the next step is bounded by $Ce^{-\alpha_1|x|}$.

Now follow self-evident steps.

$$P_{x,x}[\ Y_k \neq \bar{Y}_k \text{ for some } 1 \leq k \leq \zeta\ ]$$

$$\leq \sum_{k=1}^{\infty} P_{x,x}[\ Y_j = \bar{Y}_j \in A^c \text{ for } 1 \leq j < k,\ Y_k \neq \bar{Y}_k\ ]$$

$$\leq \sum_{k=1}^{\infty} E_{x,x}\Big[\mathbb{1}\{\ Y_j = \bar{Y}_j \in A^c \text{ for } 1 \leq j < k\ \}P_{Y_{k-1},\bar{Y}_{k-1}}(Y_1 \neq \bar{Y}_1)\Big]$$

$$\leq C \sum_{k=1}^{\infty} E_{x,x}\Big[\mathbb{1}\{\ Y_j = \bar{Y}_j \in A^c \text{ for } 1 \leq j < k\ \}e^{-\alpha_1|\bar{Y}_{k-1}|}\Big]$$

$$\leq C E_x \sum_{m=0}^{\zeta-1} e^{-\alpha_1|\bar{Y}_m|}. \hspace{3cm} \square$$

For the remainder of this section Y and $\bar{Y}$ are always coupled in the manner that satisfies Lemma A.1.

LEMMA A.2. *Let $j \in \{1, \dots, d\}$ be such that the one-dimensional random walk $\bar{Y}^j$ is not degenerate. Let r_0 be a positive integer and $\bar{w} = \inf\{n \geq 1 : \bar{Y}_n^j \leq r_0\}$ the first time the random walk $\bar{Y}$ enters the half-space $\mathcal{H} = \{x : x^j \leq r_0\}$. Couple Y and $\bar{Y}$ starting from a common initial state $x \notin \mathcal{H}$. Then there is a constant C independent of r_0 such that*

$$\sup_{x \notin \mathcal{H}} P_{x,x}[\ Y_k \neq \bar{Y}_k \text{ for some } k \in \{1, \dots, \bar{w}\}\ ] \leq C e^{-\alpha_1 r_0} \quad \text{for all } r_0 \geq 1.$$

The same result holds for $\mathcal{H} = \{x : x^j \geq -r_0\}$.

Proof. By Lemma A.1

$$P_{x,x}[\ Y_k \neq \bar{Y}_k \text{ for some } k \in \{1, \dots, \bar{w}\}\ ] \leq C E_x\Bigg[\sum_{k=0}^{\bar{w}-1} e^{-\alpha_1|\bar{Y}_k|}\Bigg]$$

$$\leq C E_{x^j}\Bigg[\sum_{k=0}^{\bar{w}-1} e^{-\alpha_1 \bar{Y}_k^j}\Bigg] = C \sum_{t=r_0+1}^{\infty} e^{-\alpha_1 t} g(x^j, t)$$

where for $s, t \in [r_0 + 1, \infty)$

$$g(s,t) = \sum_{n=0}^{\infty} P_s[\bar{Y}_n^j = t\,,\ \bar{w} > n]$$

is the Green function of the half-line $(-\infty, r_0]$ for the one-dimensional random walk $\bar{Y}^j$. This is the expected number of visits to t before entering $(-\infty, r_0]$, defined on p. 209 in Spitzer [13]. The development in Sections 18 and 19 in [13] gives the bound

$$g(s,t) \leq C(1 + (s - r_0 - 1) \wedge (t - r_0 - 1)) \leq C(t - r_0), \quad s, t \in [r_0 + 1, \infty). \quad \text{(A.3)}$$

Here is some more detail. Shift $r_0 + 1$ to the origin to match the setting in [13]. Then P19.3 on p. 209 gives

$$g(x, y) = \sum_{n=0}^{x \wedge y} u(x - n)v(y - n) \qquad \text{for } x, y > 0$$

where the functions u and v are defined on p. 201. For a symmetric random walk $u = v$. P18.7 on p. 202 implies that

$$v(m) = \frac{1}{\sqrt{c}} \sum_{k=0}^{\infty} \mathbf{P}[\mathbf{Z}_1 + \cdots + \mathbf{Z}_k = m]$$

where c is a certain constant and $\{\mathbf{Z}_i\}$ are i.i.d. strictly positive, integer-valued ladder variables for the underlying random walk. Now one can show inductively that $v(m) \leq v(0)$ for each m so the quantities $u(m) = v(m)$ are bounded. This justifies (A.3).

Continuing from further above we get the estimate claimed in the statement:

$$E_x \left[\sum_{k=0}^{\bar{w}-1} e^{-\alpha_1 |\bar{Y}_k|} \right] \leq C \sum_{t > r_0} (t - r_0) e^{-\alpha_1 t} \leq C e^{-\alpha_1 r_0}. \qquad \square$$

For the next lemmas abbreviate $B_r = [-r, r]^d$ for d-dimensional centered cubes.

LEMMA A.3. *With α_1 given in the coupling hypothesis (A.iii), fix any positive constant $\kappa_1 > 2\alpha_1^{-1}$. Consider large positive integers r_0 and r that satisfy*

$$2\alpha_1^{-1} \log r \leq r_0 \leq \kappa_1 \log r < r.$$

Then there exist a positive integer m_0 and a constant $0 < \alpha_3 < \infty$ such that, for large enough r,

$$\inf_{x \in B_r \setminus B_{r_0}} P_x[\text{without entering } B_{r_0} \text{ chain } Y \text{ exits } B_r \text{ by time } r^{m_0}] \geq \frac{\alpha_3}{r}. \qquad \text{(A.4)}$$

Proof. We consider first the case where $x \in B_r \setminus B_{r_0}$ has a coordinate x^j that satisfies $x^j \in [-r, -r_0 - 1] \cup [r_0 + 1, r]$ and $\bar{Y}^j$ is nondegenerate. For this case we can take $m_0 = 4$. A higher m_0 may be needed to move a suitable coordinate out of the interval $[-r_0, r_0]$. This is done in the second step of the proof.

The same argument works for both $x^j \in [-r, -r_0 - 1]$ and $x^j \in [r_0 + 1, r]$. We treat the case $x^j \in [r_0 + 1, r]$. One way to realize the event in (A.4) is this: starting at x^j, the $\bar{Y}^j$ walk exits $[r_0 + 1, r]$ by time r^4 through the right boundary into $[r + 1, \infty)$, and Y and $\bar{Y}$ stay coupled together throughout this time. Let $\bar{\zeta}$ be the time $\bar{Y}^j$ exits $[r_0 + 1, r]$ and $\bar{w}$ the time $\bar{Y}^j$ enters $(-\infty, r_0]$. Then $\bar{w} \geq \bar{\zeta}$. Thus the complementary probability of (A.4) is bounded by

$$\begin{aligned}
& P_{x^j}\{ \bar{Y}^j \text{ exits } [r_0 + 1, r] \text{ into } (-\infty, r_0] \} \\
& \quad + P_{x^j}\{\bar{\zeta} > r^4\} + P_{x,x}\{ Y_k \neq \bar{Y}_k \text{ for some } k \in \{1, \ldots, \bar{w}\} \}.
\end{aligned} \qquad \text{(A.5)}$$

We treat the terms one at a time. From the development on p. 253-255 in [13] we get the bound

$$P_{x^j}\{\ \bar{Y}^j \text{ exits } [r_0+1,r] \text{ into } (-\infty,r_0]\ \} \leq 1 - \frac{\alpha_4}{r} \qquad \text{(A.6)}$$

for some constant $\alpha_4 > 0$. In some more detail: P22.7 on p. 253, the inequality in the third display of p. 255, and the third moment assumption on the steps of $\bar{Y}$ give a lower bound

$$P_{x^j}\{\ \bar{Y}^j \text{ exits } [r_0+1,r] \text{ into } [r+1,\infty)\ \} \geq \frac{x^j - r_0 - 1 - c_1}{r - r_0 - 1} \qquad \text{(A.7)}$$

for the probability of exiting to the right. Here c_1 is a constant that comes from the term denoted in [13] by $M\sum_{s=0}^{N}(1+s)a(s)$ whose finiteness follows from the third moment assumption. The text on p. 254-255 suggests that these steps need the aperiodicity assumption. This need for aperiodicity can be traced back via P22.5 to P22.4 which is used to assert the boundedness of $u(x)$ and $v(x)$. But as we observed above in the derivation of (A.3) boundedness of $u(x)$ and $v(x)$ is true without any additional assumptions.

To go forward from (A.7) fix any $m > c_1$ so that the numerator above is positive for $x^j = r_0 + 1 + m$. The probability in (A.7) is minimized at $x^j = r_0 + 1$, and from $x^j = r_0 + 1$ there is a fixed positive probability θ to take m steps to the right to get past the point $x^j = r_0 + 1 + m$. Thus for all $x^j \in [r_0+1,r]$ we get the lower bound

$$P_{x^j}\{\ \bar{Y}^j \text{ exits } [r_0+1,r] \text{ into } [r+1,\infty)\ \} \geq \frac{\theta(m-c_1)}{r - r_0 - 1} \geq \frac{\alpha_4}{r}$$

and (A.6) is verified.

As in (A.3) let $g(s,t)$ be the Green function of the random walk $\bar{Y}^j$ for the half-line $(-\infty,r_0]$, and let $\tilde{g}(s,t)$ be the Green function for the complement of the interval $[r_0+1,r]$. Then $\tilde{g}(s,t) \leq g(s,t)$, and by (A.3) we get this moment bound:

$$E_{x^j}[\bar{\zeta}] = \sum_{t=r_0+1}^{r} \tilde{g}(x^j,t) \leq \sum_{t=r_0+1}^{r} g(x^j,t) \leq Cr^2.$$

Consequently, uniformly over $x^j \in [r_0+1,r]$,

$$P_{x^j}[\bar{\zeta} > r^4] \leq \frac{C}{r^2}. \qquad \text{(A.8)}$$

From Lemma A.2

$$P_x[\ Y_k \neq \bar{Y}_k \text{ for some } k \in \{1,\dots,\bar{w}\}\ ] \leq Ce^{-\alpha_1 r_0}. \qquad \text{(A.9)}$$

Putting bounds (A.6), (A.8) and (A.9) together gives an upper bound of

$$1 - \frac{\alpha_4}{r} + \frac{C}{r^2} + Ce^{-\alpha_1 r_0}$$

for the sum in (A.5) which bounds the complement of the probability in (A.4). By assumption $r_0 > 2\alpha_1^{-1}\log r$, so for large enough r the sum above is not more than $1 - \alpha_3/r$ for some constant $\alpha_3 > 0$.

The lemma is now proved for those $x \in B_r \setminus B_{r_0}$ for which some

$$j \in J \equiv \{1 \le j \le d : \text{ the one-dimensional walk } \bar{Y}^j \text{ is nondegenerate}\}$$

satisfies $x^j \in [-r, -r_0 - 1] \cup [r_0 + 1, r]$. Now suppose $x \in B_r \setminus B_{r_0}$ but all $j \in J$ satisfy $x^j \in [-r_0, r_0]$. Let

$$T = \inf\{n \ge 1 : Y_n^j \notin [-r_0, r_0] \text{ for some } j \in J\}.$$

The first part of the proof gives P_x-almost surely

$$P_{Y_T}[\text{without entering } B_{r_0} \text{ chain } Y \text{ exits } B_r \text{ by time } r^4/2] \ge \frac{\alpha_3}{r}.$$

Replacing r^4 by $r^4/2$ only affects the constant in (A.8). It can of course happen that $Y_T \notin B_r$ but then we interpret the above probability as one.

By the Markov property it remains to show that for a suitable m_0

$$\inf\big\{\, P_x[T \le r^{m_0}/2] : x \in B_r \setminus B_{r_0} \text{ but } x^j \in [-r_0, r_0] \text{ for all } j \in J \,\big\} \qquad \text{(A.10)}$$

is bounded below by a positive constant. Hypothesis (A.1) implies that for some constant b_1, $E_x T \le b_1^{r_0}$ uniformly over the relevant x. This is because one way to realize T is to wait until some coordinate Y^j takes $2r_0$ successive identical steps. By hypothesis (A.1) this random time is stochastically bounded by a geometrically distributed random variable.

It is also necessary for this argument that during time $[0, T]$ the chain Y does not enter B_{r_0}. Indeed, under the present assumptions the chain never enters B_{r_0}. This is because for $x \in B_r \setminus B_{r_0}$ some coordinate i must satisfy $x^i \in [-r, -r_0 - 1] \cup [r_0 + 1, r]$. But now this coordinate $i \notin J$, and so by hypothesis (A.ii) the one-dimensional process Y^i is constant, $Y_n^i = x^i \notin [-r_0, r_0]$ for all n.

Finally, the required positive lower bound for (A.10) comes by Chebychev. Take $m_0 \ge \kappa_1 \log b_1 + 1$ where κ_1 comes from the assumptions of the lemma. Then, by the hypothesis $r_0 \le \kappa_1 \log r$,

$$P_x[T > r^{m_0}/2] \le 2r^{-m_0} b_1^{r_0} \le 2r^{\kappa_1 \log b_1 - m_0} \le \tfrac{1}{2}$$

for $r \ge 4$. $\square$

We come to one of the main auxiliary lemmas of this development.

LEMMA A.4. *Let $U = \inf\{n \ge 0 : Y_n \notin B_r\}$ be the first exit time from B_r for the Markov chain Y. Then there exist finite positive constants C_1, m_1 such that*

$$\sup_{x \in B_r} E_x(U) \le C_1 r^{m_1} \quad \text{for all } 1 \le r < \infty.$$

Proof. First observe that $\sup_{x \in B_r} E_x(U) < \infty$ by assumption (A.1) because by a geometric time some coordinate Y^j has experienced $2r$ identical steps in succession. Throughout, let $r_0 < r$ satisfy the assumptions of Lemma A.3. Once the statement is proved for large enough r, we obtain it for all $r \geq 1$ by increasing C_1.

Let $0 = T_0 = S_0 \leq T_1 \leq S_1 \leq T_2 \leq \cdots$ be the successive exit and entrance times into B_{r_0}. Precisely, for $i \geq 1$ as long as $S_{i-1} < \infty$

$$T_i = \inf\{n \geq S_{i-1} : Y_n \notin B_{r_0}\} \quad \text{and} \quad S_i = \inf\{n \geq T_i : Y_n \in B_{r_0}\}$$

Once $S_i = \infty$ then we set $T_j = S_j = \infty$ for all $j > i$. If $Y_0 \in B_r \smallsetminus B_{r_0}$ then also $T_1 = 0$. Again by assumption (A.1) (and as observed in the proof of Lemma A.3) there is a constant $0 < b_1 < \infty$ such that

$$\sup_{x \in B_{r_0}} E_x[T_1] \leq b_1^{r_0}. \tag{A.11}$$

So a priori T_1 is finite but $S_1 = \infty$ is possible. Since $T_1 \leq U < \infty$ we can decompose as follows:

$$\begin{aligned}
E_x[U] &= \sum_{j=1}^{\infty} E_x[U, \, T_j \leq U < S_j] \\
&= \sum_{j=1}^{\infty} E_x[T_j, \, T_j \leq U < S_j] + \sum_{j=1}^{\infty} E_x[U - T_j, \, T_j \leq U < S_j].
\end{aligned} \tag{A.12}$$

We first treat the last sum in (A.12). By an inductive application of Lemma A.3, for any $z \in B_r \smallsetminus B_{r_0}$,

$$P_z[U > jr^{m_0}, \, U < S_1] \leq P_z[\, Y_k \in B_r \smallsetminus B_{r_0} \text{ for } k \leq jr^{m_0} \,]$$

$$= E_z\big[\mathbb{1}\{\, Y_k \in B_r \smallsetminus B_{r_0} \text{ for } k \leq (j-1)r^{m_0} \,\} P_{Y_{(j-1)r^{m_0}}}\{\, Y_k \in B_r \smallsetminus B_{r_0} \text{ for } k \leq r^{m_0} \,\}\big]$$

$$\leq \cdots \leq (1 - \alpha_3 r^{-1})^j.$$

Utilizing this, still for $z \in B_r \smallsetminus B_{r_0}$,

$$\begin{aligned}
E_z[U, \, U < S_1] &= \sum_{m=0}^{\infty} P_z[U > m, \, U < S_1] \\
&\leq r^{m_0} \sum_{j=0}^{\infty} P_z[U > jr^{m_0}, \, U < S_1] \leq r^{m_0+1} \alpha_3^{-1}.
\end{aligned} \tag{A.13}$$

Next we take into consideration the failure to exit B_r during the earlier excursions in $B_r \smallsetminus B_{r_0}$. Let

$$H_i = \{Y_n \in B_r \text{ for } T_i \leq n < S_i\}$$

be the event that in between the ith exit from B_{r_0} and entrance back into B_{r_0} the chain Y does not exit B_r. We shall repeatedly use this consequence of Lemma A.3:

$$\text{for } i \geq 1, \text{ on the event } \{T_i < \infty\}, \, P_x[H_i \,|\, \mathcal{F}_{T_i}] \leq 1 - \alpha_3 r^{-1}. \tag{A.14}$$

Here is the first instance.

$$E_x[U - T_j \, , \, T_j \leq U < S_j] = E_x\Big[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_j < \infty\} \cdot E_{Y_{T_j}}(U, \, U < S_1) \Big]$$

$$\leq r^{m_0+1}\alpha_3^{-1} E_x\Big[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_{j-1} < \infty\} \Big] \leq r^{m_0+1}\alpha_3^{-1}(1 - \alpha_3 r^{-1})^{j-1}.$$

Note that if Y_{T_j} above lies outside B_r then $E_{Y_{T_j}}(U) = 0$. In the other case $Y_{T_j} \in B_r \setminus B_{r_0}$ and (A.13) applies. So for the last sum in (A.12):

$$\sum_{j=1}^{\infty} E_x[U - T_j \, , \, T_j \leq U < S_j] \leq \sum_{j=1}^{\infty} r^{m_0+1}\alpha_3^{-1}(1 - \alpha_3 r^{-1})^{j-1} \leq r^{m_0+2}\alpha_3^{-2}. \quad \text{(A.15)}$$

We turn to the second-last sum in (A.12). Utilizing (A.11) and (A.14),

$$E_x[T_j \, , \, T_j \leq U < S_j] \leq \sum_{i=0}^{j-1} E_x\Big[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_j < \infty\} \cdot (T_{i+1} - T_i) \Big]$$

$$\leq b_1^{r_0}(1 - \alpha_3 r^{-1})^{j-1} \qquad\qquad\qquad\qquad\qquad\qquad \text{(A.16)}$$

$$+ \sum_{i=1}^{j-1} E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot (T_{i+1} - T_i)\mathbb{1}_{H_i} \cdot \mathbb{1}\{T_{i+1} < \infty\} \Big](1 - \alpha_3 r^{-1})^{j-1-i}.$$

Split the last expectation as

$$E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot (T_{i+1} - T_i)\mathbb{1}_{H_i} \cdot \mathbb{1}\{T_{i+1} < \infty\} \Big]$$

$$\leq E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot (T_{i+1} - S_i)\mathbb{1}_{H_i} \cdot \mathbb{1}\{S_i < \infty\} \Big]$$

$$+ \; E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot (S_i - T_i)\mathbb{1}_{H_i} \cdot \mathbb{1}\{T_i < \infty\} \Big]$$

$$\leq E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{S_i < \infty\} \cdot E_{Y_{S_i}}(T_1) \Big] + E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_i < \infty\} \cdot E_{Y_{T_i}}(S_1 \cdot \mathbb{1}_{H_1}) \Big]$$

$$\leq E_x\Big[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_{i-1} < \infty\} \Big](b_1^{r_0} + r^{m_0+1}\alpha_3^{-1})$$

$$\leq (1 - \alpha_3 r^{-1})^{i-1}(b_1^{r_0} + r^{m_0+1}\alpha_3^{-1}). \qquad\qquad\qquad\qquad \text{(A.17)}$$

In the second-last inequality above, before applying (A.14) to the H_k's, $E_{Y_{S_i}}(T_1) \leq b_1^{r_0}$ comes from (A.11). The other expectation is estimated again by iterating Lemma

A.3 and again with $z \in B_r \setminus B_{r_0}$:

$$E_z(S_1 \cdot \mathbb{1}_{H_1}) = \sum_{m=0}^{\infty} P_z[S_1 > m, H_1] \leq \sum_{m=0}^{\infty} P_z[Y_k \in B_r \setminus B_{r_0} \text{ for } k \leq m]$$

$$\leq r^{m_0} \sum_{j=0}^{\infty} P_z[Y_k \in B_r \setminus B_{r_0} \text{ for } k \leq jr^{m_0}] \leq r^{m_0+1}\alpha_3^{-1}.$$

Insert the bound from line (A.17) back up into (A.16) to get the bound

$$E_x[T_j, T_j \leq U < S_j] \leq (2b_1^{r_0} + r^{m_0+1}\alpha_3^{-1})j(1 - \alpha_3 r^{-1})^{j-2}.$$

Finally, bound the second-last sum in (A.12):

$$\sum_{j=1}^{\infty} E_x[T_j, T_j \leq U < S_j] \leq \left(2b_1^{r_0}r^2\alpha_3^{-2} + r^{m_0+3}\alpha_3^{-3}\right)(1 - \alpha_3 r^{-1})^{-1}.$$

Taking r large enough so that $\alpha_3 r^{-1} < 1/2$ and combining this with (A.12) and (A.15) gives

$$E_x[U] \leq r^{m_0+2}\alpha_1^{-2} + 4b_1^{r_0}r^2\alpha_3^{-2} + 2r^{m_0+3}\alpha_3^{-3}.$$

Since $r_0 \leq \kappa_1 \log r$ for some constant C, the above bound simplifies to $C_1 r^{m_1}$. $\square$

For the remainder of the proof we work with $B = B_r$ for $r = c_1 \log n$. The above estimate gives us one part of the argument for (A.2), namely that the Markov chain Y exits $B = [-c_1 \log n, c_1 \log n]^d$ fast enough.

Let $0 = V_0 < U_1 < V_1 < U_2 < V_2 < \cdots$ be the successive entrance times V_i into B and exit times U_i from B for the Markov chain Y, assuming that $Y_0 = z \in B$. It is possible that some $V_i = \infty$. But if $V_i < \infty$ then also $U_{i+1} < \infty$ due to assumption (A.1), as already observed. The time intervals spent in B are $[V_i, U_{i+1})$ each of length at least 1. Thus, by applying Lemma A.4,

$$\sum_{k=0}^{n-1} P_z(Y_k \in B) \leq \sum_{i=0}^{n} E_z\big[(U_{i+1} - V_i)\mathbb{1}\{V_i \leq n\}\big]$$

$$\leq \sum_{i=0}^{n} E_z\big[E_{Y_{V_i}}(U_1)\mathbb{1}\{V_i \leq n\}\big] \qquad \text{(A.18)}$$

$$\leq C(\log n)^{m_1} E_z\bigg[\sum_{i=0}^{n} \mathbb{1}\{V_i \leq n\}\bigg].$$

Next we bound the expected number of returns to B by the number of excursions outside B that fit in a time of length n:

$$E_z\left[\sum_{i=0}^{n}\mathbb{1}\{V_i \leq n\}\right] = E_z\left[\sum_{i=0}^{n}\mathbb{1}\left\{\sum_{j=1}^{i}(V_j - V_{j-1}) \leq n\right\}\right]$$

$$\leq E_z\left[\sum_{i=0}^{n}\mathbb{1}\left\{\sum_{j=1}^{i}(V_j - U_j) \leq n\right\}\right] \qquad \text{(A.19)}$$

According to the usual notion of stochastic dominance, the random vector $(\xi_1, \ldots, \xi_n)$ dominates $(\eta_1, \ldots, \eta_n)$ if

$$Ef(\xi_1, \ldots, \xi_n) \geq Ef(\eta_1, \ldots, \eta_n)$$

for any function f that is coordinatewise nondecreasing. If the $\{\xi_i : 1 \leq i \leq n\}$ are adapted to the filtration $\{\mathcal{G}_i : 1 \leq i \leq n\}$, and $P[\xi_i > a | \mathcal{G}_{i-1}] \geq 1 - F(a)$ for some distribution function F, then the $\{\eta_i\}$ can be taken i.i.d. F-distributed.

LEMMA A.5. *There exist positive constants c_1, c_2 and γ such that the following holds: the excursion lengths $\{V_j - U_j : 1 \leq j \leq n\}$ stochastically dominate i.i.d. variables $\{\eta_j\}$ whose common distribution satisfies $\mathbf{P}[\eta \geq a] \geq c_1 a^{-1/2}$ for $1 \leq a \leq c_2 n^{\gamma}$.*

Proof. Since $P_z[V_j - U_j \geq a | \mathcal{F}_{U_j}] = P_{Y_{U_j}}[V \geq a]$ where V means first entrance time into B, we shall bound $P_x[V \geq a]$ below uniformly over

$$\left\{x \notin B : \sum_{z \in B} P_z[Y_{U_1} = x] > 0\right\}.$$

Fix such an x and an index $1 \leq j \leq d$ such that $x^j \notin [-r, r]$. Since the coordinate Y^j can move out of $[-r, r]$, this coordinate is not degenerate, and hence by assumption (A.ii) the random walk $\bar{Y}^j$ is nondegenerate. As before we work through the case $x^j > r$ because the argument for the other case $x^j < -r$ is the same.

Let $\bar{w} = \inf\{n \geq 1 : \bar{Y}_n^j \leq r\}$ be the first time the one-dimensional random walk $\bar{Y}^j$ enters the half-line $(-\infty, r]$. If both Y and $\bar{Y}$ start at x and stay coupled together until time $\bar{w}$, then $V \geq \bar{w}$. This way we bound V from below. Since the random walk is symmetric and can be translated, we can move the origin to x^j and use classic results about the first entrance time into the left half-line, $\bar{T} = \inf\{n \geq 1 : \bar{Y}_n^j < 0\}$. Thus

$$P_{x^j}[\bar{w} \geq a] \geq P_{r+1}[\bar{w} \geq a] = P_0[\bar{T} \geq a] \geq \frac{\alpha_5}{\sqrt{a}}$$

for a constant α_5. The last inequality follows for one-dimensional symmetric walks from basic random walk theory. For example, combine equation (7) on p. 185 of [13] with a Tauberian theorem such as Theorem 5 on p. 447 of Feller [7]. Or see directly Theorem 1a on p. 415 of [7].

Now start both Y and $\bar{Y}$ from x. Apply Lemma A.2 and recall that $r = c_1 \log n$.

$$
\begin{aligned}
P_x[V \geq a] &\geq P_{x,x}[V \geq a, \ Y_k = \bar{Y}_k \text{ for } k = 1, \ldots, \bar{w} \ ] \\
&\geq P_{x,x}[\bar{w} \geq a, \ Y_k = \bar{Y}_k \text{ for } k = 1, \ldots, \bar{w} \ ] \\
&\geq P_{x^j}[\bar{w} \geq a] - P_{x,x}[\ Y_k \neq \bar{Y}_k \text{ for some } k \in \{1, \ldots, \bar{w}\} \ ] \\
&\geq \frac{\alpha_5}{\sqrt{a}} - Cn^{-c_1\alpha_1}.
\end{aligned}
$$

This gives a lower bound

$$
P_x[V \geq a] \geq \frac{\alpha_5}{2\sqrt{a}}
$$

if $a \leq \alpha_5^2 (2C)^{-2} n^{2c_1\alpha_1}$. This lower bound is independent of x. We have proved the lemma. $\qquad\square$

We can assume that the random variables η_j given by the lemma satisfy $1 \leq \eta_j \leq c_2 n^\gamma$ and we can assume both $c_2, \gamma \leq 1$ because this merely weakens the result. For the renewal process determined by $\{\eta_j\}$ write

$$
S_0 = 0 \,, \ S_k = \sum_{j=1}^{k} \eta_j \,, \quad \text{and} \quad K(n) = \inf\{k : S_k > n\}
$$

for the renewal times and the number of renewals up to time n (counting the renewal $S_0 = 0$). Since the random variables are bounded, Wald's identity gives

$$
\mathbf{E}K(n) \cdot \mathbf{E}\eta = \mathbf{E}S_{K(n)} \leq n + c_2 n^\gamma \leq 2n,
$$

while

$$
\mathbf{E}\eta \geq \int_1^{c_2 n^\gamma} \frac{c_1}{\sqrt{s}} \, ds \geq c_3 n^{\gamma/2}.
$$

Together these give

$$
\mathbf{E}K(n) \leq \frac{2n}{\mathbf{E}\eta} \leq C_2 n^{1-\gamma/2}.
$$

Now we pick up the development from line (A.19). Since the negative of the function of $(V_j - U_j)_{1 \leq i \leq n}$ in the expectation on line (A.19) is nondecreasing, the stochastic domination of Lemma A.5 gives an upper bound of (A.19) in terms of the i.i.d. $\{\eta_j\}$. Then we use the renewal bound from above.

$$
\begin{aligned}
E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \leq n\} \right] &\leq E_z\left[\sum_{i=0}^{n} \mathbb{1}\left\{ \sum_{j=1}^{i}(V_j - U_j) \leq n \right\} \right] \\
&\leq \mathbf{E}\left[\sum_{i=0}^{n} \mathbb{1}\left\{ \sum_{j=1}^{i} \eta_j \leq n \right\} \right] = \mathbf{E}K(n) \leq C_2 n^{1-\gamma/2}.
\end{aligned}
$$

Returning back to (A.18) to collect the bounds, we have shown that

$$\sum_{k=0}^{n-1} P_z(Y_k \in B) \leq C(\log n)^{m_1} E_z\left[\sum_{i=0}^{n} \mathbb{I}\{V_i \leq n\}\right] \leq C(\log n)^{m_1} C_2 n^{1-\gamma/2}$$

and thereby verified (A.2).

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